THERMODYNAMICS OF SOME NON-UNIFORMLY HYPERBOLIC ATTRACTORS

AGNIESZKA ZELEROWICZ

Abstract. We study thermodynamic formalism for certain dissipative maps, that is maps with non-uniformly hyperbolic attractors, which are obtained from uniformly hyperbolic systems by the slow down procedure. Namely, starting with a hyperbolic local diffeomorphism $f : U \to M$ with an attractor Λ, one slows down trajectories in a small neighborhood of a hyperbolic fixed point $p \in \Lambda$ obtaining a nonuniformly hyperbolic diffeomorphism $g : U \to M$ with a topological attractor Λ_g. We establish the existence of equilibrium measures for the family of *geometric t-potentials* defined by $\varphi_t(x) := -t\log|Dg_{|E^u(x)}|$. We identify equilibrium measures for $t = 1$. Under additional restrictions we prove the existence of $t_0 < 0$ such that the equilibrium measures are unique for every $t \neq 1$ that belongs to the interval (t_0, ∞). We show that for $t \in (t_0, 1)$ the equilibrium measures have exponential decay of correlations and satisfy the Central Limit Theorem. Our results apply to any diffeomorphism which is a small perturbation of the classical Smale-Williams Solenoid, thus providing examples of nonuniformly hyperbolic dissipative maps on a three-dimensional manifold for which one can build thermodynamic formalism.

Introduction

Given a continuous map f of a compact metric space X and a continuous potential function φ, an invariant Borel probability measure μ_φ is called an *equilibrium measure* if

$$h_{\mu_\varphi}(f) + \int_X \varphi d\mu_\varphi = \sup_{\mu \in Erg(X)} \left\{ h_\mu(f) + \int_X \varphi d\mu \right\},$$

where $Erg(X)$ denotes the set of all f-invariant ergodic Borel probability measures. By the Variational Principle, the supremum on the right hand side coincides with the topological pressure $P(\varphi)$ of the function φ.

Consider the following special case. Let M be a compact smooth manifold, $U \subset M$ an open set, $f : U \to M$ a (local) $C^{1+\alpha}$-diffeomorphism such that $\overline{f(U)} \subset U$. Let $\Lambda \subset U$ be a hyperbolic attractor for f. Assume that $f_{|\Lambda}$ is topologically transitive. Then it is known that for every Hölder continuous potential φ, there exists a unique equilibrium measure μ_φ. In addition, μ_φ is Bernoulli, has exponential decay of correlations and satisfies the Central Limit Theorem (see [2]). Of special interest for this group of maps is the family of *geometric t-potentials* defined by $\varphi_t(x) := -t\log|Df_{|E^u(x)}|$, where t is a real number and $E^u(x)$ is the unstable subspace at x. Since $E^u(x)$ depends Hölder continuously on x, for each t the potential φ_t is Hölder continuous and hence, admits a unique equilibrium measure μ_t. It is also known that the pressure function $P(t) := P(\varphi_t)$ is convex, decreasing and real analytic in t.

Date: October 19, 2018.

The author is partially supported by NSF grants 754911, 1400027 and 1101165.

Away from uniform hyperbolicity there are classes of dynamical systems for which thermodynamic formalism is rather well understood. In particular, existence, uniqueness, and statistical properties of equilibrium measures for the geometric t-potential are established for some intervals in t. Moreover, phase transitions where the pressure function is non-differentiable are found. An example of such dynamical system, which is the most closely related to this paper, is the Katok map introduced in [7] by Katok. Katok's construction starts with a linear hyperbolic automorphism A of the 2-torus and proceeds by slowing down trajectories in a small neighborhood of a hyperbolic fixed point. As a result this point becomes neutral thus producing trajectories with zero Lyapunov exponents. Thermodynamics of the Katok map was studied by Pesin, Senti and Zhang in [10]. The approach was based on showing that the Katok map is a Young diffeomorphism (see Section 4 for the definition) and then applying the results from their earlier paper, [9].

The goal of this paper is to effect thermodynamic formalism for certain dissipative maps, that is maps with non-uniformly hyperbolic attractors, which are obtained from uniformly hyperbolic systems by the slow down procedure. Namely, starting with a hyperbolic local diffeomorphism $f : U \to M$ with an attractor Λ, one slows down trajectories in a small neighborhood of a hyperbolic fixed point $p \in \Lambda$ obtaining a nonuniformly hyperbolic diffeomorphism $g : U \to M$ with a topological attractor Λ_g (see Section 1 for details).

A major part of the paper is devoted to describing certain hyperbolic properties of the map g (Section 3) and ultimately presenting it as a Young diffeomorphism (Section 4). For simplicity of calculations we make some technical assumptions. We allow the manifold M to be of arbitrary dimension, however the unstable subspace for f is assumed to be one-dimensional. In addition we require some conformality in the stable direction.

Building thermodynamics for Young diffeomorphisms requires a certain, rather strong bound on the growth rate of the number of partition elements in the base of the tower (see (4.7)). This puts additional restrictions on the hyperbolic map f. We assume that f lies in a sufficiently small C^1-neighborhood of a map for which the SRB measure and the measure of maximal entropy coincide. This is the case if, for instance, $\log |J^u f(x)|$ is cohomologous to a constant. As an example one may consider the classical Smale-Williams Solenoid or any of its C^1-small perturbations.

While we follow the approach in [10], our case is quite different and requires addressing a number of highly nontrivial issues. Certain results essential for the argument in [10], which are well known for the Katok map, were not known and had to be established in our case. In addition one had to carefully handle many difficulties coming from the dissipativity of the system. We explain the main issues below.

It is known that the Katok map is topologically conjugate to the linear toral automorphism, from which it is obtained by the slow down process. From this one immediately concludes the existence of a Markov partition. The crucial step in [10] was to use an element of a Markov partition as the base of the Young tower. It is then essential to show that the map g in our construction is topologically conjugate to the hyperbolic map f. We prove this in Theorem 2 but our approach is quite different than the one in the original Katok's paper as the latter cannot be used in our case. We achieve that by using a version of the Anosov Shadowing Theorem to obtain a continuous map h that satisfies the semiconjugacy relation and is close to identity. In a way similar to [4], we show that such a map h is injective if g is expansive with sufficiently large expansivity constant. In the case of the Katok map, such an h is automatically surjective as a continuous map on the torus which is C^0-close to identity. In our setting showing surjectivity is one of the major issues caused mainly by the fact that the attractor Λ for f is not preserved under the perturbation.

Another crucial obstacle that we have to deal with is the existence of invariant families of stable and unstable embedded discs in the base of the Young tower. In the case of the Katok map it was shown in [7] by a simple argument that the conjugacy map transforms stable and unstable curves of the linear map A into smooth curves thus producing an invariant stable and unstable foliation for the Katok map. In our case, since the conjugacy map is defined only on the attractor, it does not provide much information about stable manifolds. Therefore, we use a different approach. It is based on results from [3], on existence of invariant families of stable and unstable cones for g together with expansion and contraction estimates within the cones in the perturbed area. Those results and standard cone techniques allow us to first establish existence of an invariant splitting of the tangent bundle and then to prove the existence of an invariant stable and unstable foliation for g (see Theorem 3.3).

Yet another crucial problem was to establish the bound for the distortion, which is required in the definition of the Young diffeomorphism (see Condition (A3)). The estimates for the distortion along unstable leaves in the perturbed area are shown in [3]. The proof uses the following estimate for the distance between two trajectories, $x(t)$ and $y(t)$ on the same unstable leaf, where x enters the perturbed region at $t = 0$ and exits at $t = T$:

$$d(x(t), y(t)) \leq C(T + 1 - t)^{-(1+\frac{1}{\alpha})} d(x(T), y(T)),$$

where α is the Hölder exponent of the slow-down function. Because of multidimensionality of the stable space (regardless of conformality) the analogous estimate for two trajectories on the same stable leaf are weaker (see (5.30)) and not sufficient for the rest of the argument. We therefore had to strengthen other estimates presented in [3] (see Lemma 5.7).

The structure of the paper is as follows. In Section 1 we introduce the setting. We list the assumptions on the hyperbolic map f and describe the slow down procedure. In Section 2 we present the main theorems establishing thermodynamics of the map g. We start Section 3 by stating the result about the conjugacy between the two maps. Then we state hyperbolic properties of the map g, such as existence and regularity of stable and unstable manifolds. In Section 4 we give the definition of a Young diffeomorphism and recall a general result describing its thermodynamics. We present all the proofs in Section 5.

Acknowledgments. I would like to thank The Erwin Schrödinger International Institute for Mathematics and Physics (ESI), Austria and the ICERM at Brown University where part of this work was done. I also wish to thank my advisor, Y. Pesin, for posing the problem and for valuable suggestions. I want to thank V. Climenhaga for his comments.

1. The setting

Let M be a d-dimensional, compact, smooth Riemannian manifold and $U \subset M$ an open set. Let $f : U \to M$ be a $C^{1+\alpha}$ diffeomorphism onto its image with $\overline{f(U)} \subset U$, where $\alpha \in (0, 1)$. [1]. Let $\Lambda = \bigcap_{n \geq 0} \overline{f^n(U)}$ be an attractor for f with $NW(f) = \Lambda$. Note that then necessarily Λ is compact. Assume that,

[1] All results in the paper apply if f is of class C^{1+r} for any $r > 0$. However, in the construction of the map g (see **(D0)**) one should use a number $0 < \alpha < \min\{1, r\}$

(C1) Λ is a hyperbolic set for f, so that for every $x \in \Lambda$ there exists a splitting of the tangent space, $T_xM = E^u_f(x) \oplus E^s_f(x)$, with $Df(x)(E^u_f(x)) = E^u_f(f(x))$ and $Df(x)(E^s_f(x)) = E^s_f(f(x))$ such that,

$$\begin{aligned} \|Df(x)(v^u)\| &\geq \nu\|v^u\| \quad \text{for all} \quad v^u \in E^u_f(x), \text{ and} \\ \|Df(x)(v^s)\| &\leq \nu^{-1}\|v^s\| \quad \text{for all} \quad v^s \in E^s_f(x), \end{aligned} \tag{1.1}$$

for some $\nu > 1$.

(C2) The unstable distribution $E^u_f(x)$ is one-dimensional for all $x \in \Lambda$.

(C3) The map f has a fixed point $p \in \Lambda$.

Without loss of generality we shall also assume that $f_{|\Lambda}$ is topologically transitive. In a presence of a fixed point this is equivalent to the assumption that the spectral decomposition of Λ has exactly one element and in fact $f_{|\Lambda}$ is topologically mixing. This assumption is not essential and one can easily see how the results presented in the paper extend to the general case.

Consider a neighborhood $Z_0 \subset U$ of p with local coordinates identifying the decomposition $E^u_f(p) \oplus E^s_f(p)$ with $\mathbb{R} \oplus \mathbb{R}^{d-1}$.

(C4) There exists a neighborhood $Z \subset Z_0$ of p on which f is the time-1 map of the flow generated by a linear vector field, $\dot{x} = Ax$, where $A = A_u \oplus A_s$ with $A_u = \gamma Id_u$ and $A_s = -\beta Id_s$ for some $\beta > \gamma > 0$.

From now on we use local coordinates in Z and identify p with 0. Fix $0 < r_0 < r_1$ such that $B(0, r_1) \subset Z \subset U$, and let $\psi : [0,1] \to [0,1]$ be a $C^{1+\alpha}$ function satisfying the following condition:

(D0)
- *a)* $\psi(r) = r^\alpha$ for $r \leq r_0$;
- *b)* $\psi(r) = 1$ for $r \geq r_1$;
- *c)* $\psi'(r) \geq 0$.

Let $\chi : Z \to \mathbb{R}^d$ be the vector field given by $\chi(x) = \psi(\|x\|)Ax$ and let $g : U \to M$ be the time-1 map of the flow generated by this vector field on Z and by f on $U \setminus Z$. Observe that g is of class $C^{1+\alpha}$ and that $g(U) = f(U)$, in particular $\overline{g(U)} \subset U$ and then $\Lambda_g := \bigcap_{n \geq 0} \overline{g^n(U)}$ is an attractor for g.

Let μ be an invariant ergodic hyperbolic measure for the map g, that is, all Lyapunov exponents of μ are non-zero (see [1] for definitions). Given a regular set Y_l of positive measure, and sufficiently small $r > 0$, let $Q_l(x) = \bigcup_{w \in Y_l \cap B(x,r)} V^u(w)$ for every $x \in Y_l$, where $V^u(w)$ is the local unstable manifold through w. Denote by $\xi(x)$ the partition of $Q_l(x)$ by these manifolds, and let $\mu^u(w)$ be the conditional measure on $V^u(w)$ generated by μ with respect to the partition ξ.

Definition 1.1. *A hyperbolic invariant measure μ is called an SRB measure (for Sinai-Ruelle-Bowen) if for any regular set Y_l of positive measure and almost every $x \in Y_l, w \in Y_l \cap B(x, r)$, the conditional measure $\mu^u(w)$ is absolutely continuous with respect to the leaf volume $m_{V^u(w)}$ on $V^u(w)$.*

It is known, that every *SRB* measure is a physical measure, that is, the set of generic points has the full volume.

The following result was shown in ([3], Theorem 2.4).

Proposition 1.1. *If f satisfies conditions **(C1)-(C4)** and the function ψ satisfies condition **(D0)**, then the map g has an SRB measure supported on Λ_g.*

Remark 1.1. *In fact, we will show later, Theorem B that the SRB measure is unique.*

2. Main result: thermodynamics of the map g

Recall that given a continuous function f of a compact metric space X we say that f has *exponential decay of correlations* with respect to a measure $\mu \in Erg(X)$ and a class $\mathcal{H}$ of functions if there exists $0 < \theta < 1$ such that for any $h_1, h_2 \in \mathcal{H}$,

$$|\int h_1(f^n(x))h_2(x)d\mu - \int h_1(x)d\mu \int h_2(x)d\mu| \leq K\theta^{|n|}$$

for some $K = K(h_1, h_2) > 0$ and all $n \in \mathbb{Z}$. The transformation f satisfies the *Central Limit Theorem* (CLT) with respect to a measure μ for functions in $\mathcal{H}$ if for any $h \in \mathcal{H}$, which is not a coboundary (i.e., $h \neq u \circ f - u + c$ for any $u \in \mathcal{H}$ and $c \in \mathbb{R}$), there exists $\sigma > 0$ such that

$$\mu\left\{\frac{1}{\sqrt{n}}\sum_{i=0}^{n-1}(h(f^i(x)) - \int hd\mu) < t\right\} \to \frac{1}{\sigma\sqrt{2\pi}}\int_{-\infty}^{t} e^{-\tau^2/2\sigma^2}d\tau \text{ as } n \to \infty.$$

Consider the geometric $t-$potential

$$\varphi_{t,g} = -t\log|J^u g| := -t\log|Dg_{|E^u_g(x)}|,$$

where E^u_g is defined later in Theorem 3.3. Our goal is to describe the existence, uniqueness and ergodic properties of the equilibrium measures associated to $\varphi_{t,g}$.

Theorem A. *If $r_1 > 0$ is sufficiently small, then for every $t \in \mathbb{R}$ the map g admits an equilibrium measure associated to $\varphi_{t,g}$.*

Let $\mathcal{P}(M)$ denote the collection of uniformly hyperbolic local diffeomorphisms $\tilde{f}$ on M which lie in a sufficiently small C^1-neighborhood of a map for which the SRB measure and the measure of maximal entropy coincide. The requirement on the C^1 distance between the maps is made precise in the proof of Lemma 5.13. Observe that $\mathcal{P}(M)$ contains all $\tilde{f}$ for which $\log|J^u\tilde{f}(x)|$ is cohomologous to a constant. As an example one may consider the classical Smale-Williams Solenoid or any of its C^1-small perturbations.

The following result describes the uniqueness and ergodic properties of the equilibrium measures associated to $\varphi_{t,g}$ for some values of t.

Theorem B. *Assume that f satisfies conditions **(C1)-(C4)** and the function ψ satisfies condition **(D0)** with $r_1 > 0$ small enough. Then the following statements hold with respect to g:*

- *for $t = 1$ there are two ergodic equilibrium measures, the unique SRB-measure and the Dirac measure at 0;*
- *for $t > 1$ the Dirac measure at 0 is the only equilibrium measure.*

  If in addition $f \in \mathcal{P}(M)$, then there exists a number $t_0 = t_0(r_1) < 0$, such that for any $t_0 < t < 1$ the following statements hold with respect to g:
- *there exists a unique equilibrium measure $\mu_{t,g}$;*
- *$\mu_{t,g}$ has exponential decay of correlations and satisfies CLT with respect to a class of functions, which includes all Hölder continuous functions on Λ_g;*

if in addition $|J^u f(x)|$ is a constant on Λ, then $t_0 \to -\infty$ as $r_1 \to 0$.

The proofs of theorems A and B are presented in Section 5.3. Note that Theorem A follows easily if one can prove continuity of $E^u_g(x)$ and expansivity of g. Those properties are established in the next section. The proof of Theorem B occupies the main part of the paper.

3. Hyperbolic properties of the map g

Let f and g be as in Section 1. All the results in this section are proved in Section 5.1.

The next result describes some topological properties of the map g.

Theorem 3.1. *If $r_1 > 0$ is sufficiently small, then*

(1) the map g is expansive and the expansivity constant is nonincreasing in r_1;

(2) there exists a homeomorphism $h : \Lambda_g \to \Lambda$ such that

$$h \circ g_{|\Lambda_g} = f \circ h;$$

in addition, there exists $C > 0$ such that $d_{C^0}(Id_{|\Lambda_g}, h) \leq Cr_1$.

We shall follow the approach from [3] to obtain continuous families of stable and unstable cones for the map g.

Definition 3.1. *Given $x \in M$, a subspace $E(x) \subset T_xM$, and $\rho(x) > 0$, the cone at x around $E(x)$ with angle $\rho(x)$ is defined by,*

$$K(x, E(x), \rho(x)) = \{v \in T_xM | \angle(v, E(x)) < \rho(x)\}. \tag{3.2}$$

Our goal is to find a number $\rho > 0$ not depending on x and construct two families of cones, $K^u_\rho(x) = K(x, E_1(x), \rho)$ and $K^s_\rho(x) = K(x, E_2(x), \rho)$, which are invariant in the following sense:

$$\overline{DgK^u_\rho(x)} \subset K^u_\rho(g(x)), \text{ and } \overline{K^s_\rho(g(x))} \subset DgK^s_\rho(x).$$

Proposition 3.2. *There exist $0 < \rho < \frac{\pi}{4}$ and continuous families of invariant cones, $K^u_\rho(x)$ and $K^s_\rho(x)$, defined on U with the following properties. There are constants $C > 0$ and $\nu > 1$ such that:*

- *for every $v \in K^u_\rho(x)$:*
  a) if $x, g(x), \ldots g^n(x) \notin B(0, r_1)$ then,

$$\|Dg^n(x)(v)\| \geq C\nu^n \|v\|; \tag{3.3}$$

  b) if $x \notin B(0, r_1)$ and $g(x) \in B(0, r_1)$ then,

$$\|Dg^k(x)(v)\| \geq C\|v\|, \tag{3.4}$$

  where $k > 0$ is the smallest integer such that $g^k(x) \notin B(0, r_1)$;
- *for every $v \in K^s_\rho(x)$,*
  a) if $x, g^{-1}(x), \ldots g^{-n}(x) \in U \setminus B(0, r_1)$ then,

$$\|Dg^{-n}(x)(v)\| \geq C\nu^n \|v\|; \tag{3.5}$$

  b) if $x \notin B(0, r_1)$ and $g^{-1}(x) \in B(0, r_1)$ then,

$$\|Dg^{-k}(x)(v)\| \geq C\|v\|, \tag{3.6}$$

  where $k > 0$ is the smallest integer such that $g^{-k}(x) \notin B(0, r_1)$.

Remark 3.1. *Throughout the paper we will often say that a given submanifold W is contained in a stable (or unstable) cone and write $TW \subset K^s_\rho$ (respectively $TW \subset K^u_\rho$). By this we mean that for every point $x \in W$ we have that $T_xW \subset K^s_\rho(x)$ (respectively $T_xW \subset K^u_\rho(x)$).*

Using the families of cones, $K^s_\rho(x)$ and $K^u_\rho(x)$, we establish the existence of invariant subspaces and admissible manifolds for g at every point on Λ_g. For this we need to define the notion of $C^{1+\alpha}$ submanifolds. Consider $W \subset U$, a C^1 embedded disc of co-dimension 1, and a point $x \in W$. In local coordinates for x we can identify $E^s_f(x)$ with $\mathbb{R}^{d-1}$ and $E^u_f(x)$ with $\mathbb{R}$. If $T_xW \subset K^s_\rho(x)$, then T_xW can be viewed as a graph of a linear function $A_x : \mathbb{R}^{d-1} \to \mathbb{R}$. The same is true for every $y \in W$ in the same chart. We say that an embedded disc $W \subset U$ with $TW \subset K^s_\rho$ is $C^{1+\alpha}$ if it is C^1 and there exists $L > 0$ such that for any $x, y \in W$ as above,

$$\|A_x - A_y\| \leq L\|x - y\|^\alpha.$$

Similarly, we define $C^{1+\alpha}$ discs in the unstable cones.

Theorem 3.3. *If $r_1 > 0$ is sufficiently small, then*

1. *For every $x \in \Lambda_g$ except $x = 0$, the intersection*
$$E^u_g(x) := \bigcap_{n\geq 0} Dg^n K^u_\rho(g^{-n}(x))$$
is a one dimensional subspace of T_xM.
2. *For every $x \in \Lambda_g$ except $x = 0$, the intersection*
$$E^s_g(x) := \bigcap_{n\geq 0} Dg^{-n} K^s_\rho(g^n(x))$$
is a subspace of T_xM of dimension $dimM - 1$.
3. *The subspaces $E^u_g(x)$ and $E^s_g(x)$, $x \neq 0$, depend continuously on x and can be extended by continuity to $x = 0$.*
4. *For every $x \in \Lambda_g$ there exist $C^{1+\alpha}$ embedded discs W^u_x, $W^s_x \subset U$ such that $T_xW^u_x = E^u_g(x)$, $T_xW^s_x = E^s_g(x)$. The sizes of $W^{s,u}_x$ and the Hölder constant are uniform in x. In fact W^u_x and $W^s_x \cap \Lambda_g$ are images of unstable and stable manifolds of f (intersected with Λ) under h^{-1}.*

4. The map g as a Young diffeomorphism

The maps introduced in [11] and [12] provide a wide collection of examples for which one can prove the existence of SRB measures. Thermodynamics of those maps has been studied in [9]. We give a definition of Young diffeomorphisms following [9], which is more suitable to our situation.

Let M be a compact smooth Riemannian manifold and $U \subset M$ an open set. Let $f : U \to M$ be a $C^{1+\alpha}$ diffeomorphism onto its image. Assume that $\Lambda \subset M$ is a topological attractor for f. An embedded disc $V \subset M$ is called an *unstable disc* (respectively *stable disk*) if for all $x, y \in V$ we have that $d(f^{-n}(x), f^{-n}(y)) \to 0$ (respectively $d(f^n(x), f^n(y)) \to 0$) as $n \to \infty$. A collection of embedded discs $\Gamma^u = \{V^u\}$ is called a *continuous family of C^1 unstable discs* if there exists a homeomorphism $\Theta : K^s \times D^u \to \bigcup V^u$ satisfying:

- $K^s \subset M$ is a Borel subset and $D^u \subset \mathbb{R}^d$ is the closed unit disc for some $d < dimM$;
- $x \to \Theta_{|\{x\}\times D^u}$ is a continuous map from K^s to the space of C^1 embeddings of D^u into M which can be extended to a continuous map of the closure $\overline{K^s}$;
- $V^u = \Theta(\{x\} \times D^u)$ is an unstable disc.

A *continuous family of C^1 stable discs* is defined similarly.

We say that a set $P \subset \Lambda$ has *hyperbolic product structure* if there exists a continuous family $\Gamma^u = \{V^u\}$ of unstable discs V^u and a continuous family $\Gamma^s = \{V^s\}$ of stable discs V^s such that

- $dim V^s + dim V^u = dim M$;
- the V^u-discs are transversal to V^s-discs by an angle uniformly bounded away from zero;
- each V^u-disc intersects each V^s-disc at exactly one point;
- $P = (\cup V^u) \cap (\cup V^s)$.

A subset $P_0 \subset P$ is called an *s-subset* if it has hyperbolic product structure and is defined by the same family Γ^u of unstable disks as P and a continuous subfamily $\Gamma^s_0 \subset \Gamma^s$ of stable discs. A *u-subset* is defined analogously.

The map f is called a *Young diffeomorphism* if it satisfies the following conditions:

(A0) There exists $P \subset \Lambda$ with a hyperbolic product structure.

(A1) There exists a countable collection of continuous subfamilies $\Gamma^s_i \subset \Gamma^s$ of stable discs and positive integers τ_i, $i \in \mathbb{N}$ such that the s-subsets

$$P^s_i := \bigcup_{V \in \Gamma^s_i} (V \cap P) \subset P$$

are pairwise disjoint and satisfy

(a) *invariance*: for every $x \in P^s_i$

$$f^{\tau_i}(V^s(x)) \subset V^s(f^{\tau_i}(x)),\ f^{\tau_i}(V^u(x)) \supset V^u(f^{\tau_i}(x)),$$

where $V^{u,s}$ denotes the (un)stable disc containing x;

(b) *Markov property*: $P^u_i := f^{\tau_i}(P^s_i)$ is a *u-subset* of P such that for all $x \in P^s_i$

$$f^{-\tau_i}(V^s(f^{\tau_i}(x)) \cap P^u_i) = V^s(x) \cap P,$$

$$f^{\tau_i}\,(V^u(x) \cap P^s_i) = V^u(f^{\tau_i}(x)) \cap P.$$

For any $x \in P^s_i$ define the *inducing time* by $\tau(x) := \tau_i$ and the *induced map* $F : \bigcup_{i \in \mathbb{N}} P^s_i \to P$ by

$$F_{|P^s_i} := f^{\tau_i}_{|P^s_i}.$$

(A2) There exists $0 < a < 1$ such that for any $i \in \mathbb{N}$ we have:

(a) For any $x \in P^s_i$ and $y \in V^s(x)$,

$$d(F(x), F(y)) \le a d(x, y);$$

(b) For any $x \in P^s_i$ and $y \in V^u(x) \cap P^s_i$,

$$d(x, y) \le a d(F(x), F(y)).$$

For $x \in P$ denote

$$E^u(f^k(x)) := T_{f^k(x)} f^k(V^u(x)) = Df^k T_x V^u(x)$$

and denote the restriction of the Jacobian of f to E^u by $J^u f(x) := \det Df_{|E^u(x)}$. The definition of $J^u F(x)$ is analogous.

(A3) There exists $c > 0$ and $0 < \beta < 1$ such that:

(a) For all $n \geq 0$, $x \in F^{-n}(\bigcup_{i\in\mathbb{N}} P_i^s)$ and $y \in V^s(x)$ we have

$$\log \frac{J^uF(F^nx)}{J^uF(F^ny)} \leq c\beta^n;$$

(b) For any $i_0, \dots, i_n \in \mathbb{N}$, $F^k(x), F^k(y) \in P_{i_k}^s$ for $0 \leq k \leq n$ and $y \in V^u(x)$ we have

$$\log \frac{J^uF(F^{n-k}x)}{J^uF(F^{n-k}y)} \leq c\beta^k.$$

Let m_{V^u} be the leaf volume on V^u.

(A4) For every $V^u \in \Gamma^u$ one has

$$m_{V^u}((P \setminus \cup P_i^s) \cap V^u) = 0 \ \text{ and } \ m_{V^u}(V^u \cap P) > 0.$$

(A5) There exists $V^u \in \Gamma^u$ such that

$$\sum_{i=1}^{\infty} \tau_i m_{V^u}(P_i^s) < \infty.$$

Define

$$scl(P_i^s) := \bigcup_{x \in \overline{P_i^s \cap V^u}} V^s(x) \cap P.$$

One can easily see that the definition of $scl(P_i^s)$ does not depend on the choice of V^u.

(A6) The set $\bigcup_{i\in\mathbb{N}} (scl(P_i^s) \setminus P_i^s)$ supports no invariant measure that gives a positive weight to any set which is open in the induced topology on Λ.

Remark 4.1. *Condition* $(A6)$ *is equivalent to Condition* $(I3)$ *in* [9] *and is crucial in construction of equilibrium measures.*

The following proposition is a direct corollary of Theorems 7.1 and 7.7 in [9] and Theorem B.1. in [8] applied to a special case, when τ_i in $(A1)$ is the first return time to P and the map f is topologically transitive. Denote the set of invariant ergodic measures on Λ by $Erg(\Lambda)$.

Proposition 4.1. *Let $f : U \to M$ be a $C^{1+\epsilon}$ embedding of a compact smooth Riemannian manifold M, which is topologically transitive on its topological attractor $\Lambda \subset U$. If there exists a set $P \subset \Lambda$ satisfying conditions $(A0) - (A6)$, then the following statements hold:*

(1) There exists an equilibrium measure μ_1 on Λ for the potential φ_1 which is the unique SRB measure;

(2) Assume that τ_i in $(A1)$ is the first return time to P and that

$$S_n = \#\{i \mid \tau_i = n\} \leq Ce^{hn} \tag{4.7}$$

with some $C > 0$ and $0 < h < -\int \varphi_1 d\mu_1$. Define

$$t_0 := \frac{h + \int_\Lambda \varphi_1 d\mu_1}{(\max_{x\in\Lambda} \varphi_1(x)) - \int_\Lambda \varphi_1 d\mu_1} < 0.$$

For every $t_0 < t < 1$ there exists a measure μ_t on Λ which is a unique equilibrium measure for the potential φ_t in $Erg(\Lambda)$ for which $\mu(P) > 0$, i.e.,

$$h_{\mu_t}(f) + \int_\Lambda \varphi_t d\mu_t = \sup_{\mu \in Erg(\Lambda),\ \mu(P)>0} \{h_\mu(f) + \int_\Lambda \varphi_t d\mu\}.$$

(3) *Assume in addition that the greatest common divisor (gcd) of $\{\tau_i \mid i \in \mathbb{N}\}$ is 1. Assume also that there exists $K > 0$ such that*

$$d(f^j(x), f^j(y)) \leq K \max\{d(x,y), d(F(x), F(y))\} \tag{4.8}$$

for every $i \geq 0$, $x, y \in P_i^s$ and any $0 \leq j \leq \tau_i$. Then for every $t_0 < t < 1$ the measure μ_t has exponential decay of correlations and satisfies the CLT with respect to a class of functions which contains all Hölder continuous functions on Λ.

We have the following for the map g. The proof is presented in Section 5.2.

Theorem 4.2. *The map g constructed in Section 1 is a Young diffeomorphism with τ_i in Condition $(A1)$ to be the first return time to P.*

5. Proofs

We prove results from Section 3 in Section 5.1. The proof of Theorem 4.2 occupies Section 5.2. The proofs of Theorems A and B are presented in Section 5.3.

5.1. **Proofs of results from Section 3.**

We start by proving Proposition 3.2, which is a corollary of two estimates derived in [3]. We then prove Theorems 3.1 and 3.3 in order.

Proof of Proposition 3.2. Consider a family $K_\rho^u(x)$ of ρ-cones around unstable subspaces of f defined on Λ. Since the splitting in (1.1) is continuous, it extends to a neighborhood of Λ, and so in particular we may assume without loss of generality that (1.1) continues to hold on all of U. Because of that the cones are defined for all $x \in U$. The following result is proved in ([3], Lemma 7.7 and formula (7.24)).

Lemma 5.1. *Let $x : [0,T] \to B(0, r_1)$ be the trajectory of the flow generated by $\chi := \|x\|^\alpha Ax$ and let $\theta(t)$ be the positive angle between the vector $\vec{0x}$ and $E^u(0)$. For a vector $v \in T_xM$ write $v = v_s + v_u$ with $v_s \in E^s(0)$ and $v_u \in E^u(0)$. Let ρ_u be the positive angle between v and E^u and let ρ_s be the positive angle between v and E^s. Denote $\lambda = \gamma + \beta$. We have the following:*

(1) *If $\tan \rho_u \leq 1$, then*

$$(\tan \rho_u)' \leq -\lambda \|x\|^\alpha \tan \rho_u + \alpha \lambda \|x\|^\alpha \frac{\tan \theta}{1 + \tan^2 \theta},$$

(2) *if $\tan \rho_s \leq 1$, then*

$$(\tan \rho_s)' \geq \lambda \|x\|^\alpha \tan \rho_u - \alpha \lambda \|x\|^\alpha \frac{\tan \theta}{1 + \tan^2 \theta}.$$

Noting that $r/(1 + r^2) \leq 1/2$ for all $r \in \mathbb{R}$ we conclude that if r_1 is small enough, there exist an angle $0 < \rho < \frac{\pi}{4}$ such that the family $K_\rho^u(x)$ is invariant under the dynamics of Dg. Then since $f = g$ outside of $B(0, r_1)$, by (1.1), there is $C > 0$ such that (3.3) holds. The inequality (3.4) is proved in ([3], inequality (7.42)).

In a similar manner we consider a family of stable cones, $K^s_\rho(x)$, on U. This family is backward invariant in a sense that $\overline{Dg^{-1}K^s_\rho(x)} \subset K^s_\rho(g^{-1}(x))$, if $g^{-1}(x) \in U$. Also the estimates (3.3) and (3.4) can be shown to remain true for vectors in $K^s_\rho(x)$ and the reversed time. Precise estimates leading to (3.6) are established later in (5.29). □

Remark 5.1. *Note that for any $Q > 0$ we can choose r_1 small enough so that every point exiting Z (see (C4) for definition) will spend at least Q steps in the complement of $B(0, r_1)$. We shall choose Q large enough to ensure that $C^2\nu^Q > 1$, where C is the constant from Proposition 3.2.*

Proof of Statement 1 in Theorem 3.1. Consider two nearby points $x, y \notin B(0, r_1)$ and a geodesic γ with x and y as endpoints.

We start with a case when $T\gamma \subset K^u_\rho$ (see Remark 3.1). Assume first that forward trajectories of all the points in γ are infinitely often outside of $B(0, r_1)$. Denote the length of γ by $L(\gamma)$. If for some positive integers m and l the images $g^m(\gamma), \dots, g^{m+l}(\gamma)$ are all outside of $B(0, r_1)$, then (3.3) gives that

$$L(g^{m+l}(\gamma)) \geq C\nu^l L(g^m(\gamma)). \tag{5.9}$$

On the other hand, if m' and l' are such that $g^{m'}(\gamma) \cap B(0, r_1) \neq \emptyset$ and $g^{m'+l'}(\gamma) \cap B(0, r_1) = \emptyset$, then by (3.4),

$$L(g^{m'+l'}(\gamma)) \geq CL(g^{m'}(\gamma)). \tag{5.10}$$

Define $\bar{\delta} := dist(B(0, r_1), U \setminus Z)$, where Z is defined in Condition (C4). By (5.9), (5.10) and Remark 5.1, as long as $d(g^n(x), g^n(y)) \leq \bar{\delta}$, the length of γ expands with the rate at least $(C^2\nu^Q)^k$, where k counts the number of exits of $g^n(x)$ from Z. In particular, taking large $n > 0$ we have that $d(g^n(x), g^n(y)) > \delta$ for some $\delta > 0$ not depending on x and y.

Now consider a point $p \in \gamma$ and $n_0 > 0$ such that $g^n(p) \in B(0, r_1)$ for all $n \geq n_0$. We then must have that $g^n(p) \in W^s_{0,f}$, where $W^s_{0,f}$ denotes a local stable manifold of 0 for f. Since $T\gamma \subset K^u_\rho$, the endpoints of γ eventually exit Z. In particular, for some $m \geq n$ we get that $d(g^m(x), g^m(y)) > \bar{\delta}$.

In a similar manner we study the case when $T\gamma \subset K^s_\rho$. If backward trajectories of all the points in γ are infinitely often outside of $B(0, r_1)$, then as long as x and y stay close, by (3.5) and (3.6), γ expands backwards with the rate at least $(C^2\nu^Q)^k$, where k counts the number of exits of x from $B(0, r_1)$. In particular taking large $n > 0$ we have that $d(g^{-n}(x), g^{-n}(y)) > \delta$ for some $\delta > 0$ not depending on x and y. If now there is a point $p \in \gamma$ and $n_0 > 0$ such that $g^{-n}(p) \in B(0, r_1)$ for all $n \geq n_0$, then we must have that $g^{-n}(p) \in W^u_{0,f}$. Since $T\gamma \subset K^s_\rho$, the endpoints of γ eventually exit Z. In particular for some $m \geq n$ we get that $d(g^{-m}(x), g^{-m}(y)) > \bar{\delta}$.

We now turn to a case when γ is not contained in a stable or unstable cone. Then for every vector $v \in T_p\gamma$, with $p \in \gamma$, the angle between v and the stable subspace at p (for f) is greater than $\tilde{\rho}$ for some $\tilde{\rho} < \rho$. If x and y are close enough, we can guarantee that $\tan\tilde{\rho} \geq \alpha/2$. Let v_s and v_u denote the projection of v on stable and unstable subspace correspondingly and let $\eta(n)$ denote the positive angle between $Dg^n(v)$ and the unstable space at $g^n(p)$. We then have that

$$\tan\eta(0) \leq \frac{1}{\tan\tilde{\rho}},$$

where $\tan\eta(0) = \frac{\|v_s\|}{\|v_u\|}$. By (1.1), we obtain that

$$\tan\eta(n) = \frac{\|Dg^n(v_s)\|}{\|Dg^n(v_u)\|} \leq \frac{1}{C^2}\nu^{-2n}\tan\eta(0) \leq \frac{1}{C^2\tan\tilde{\rho}}\nu^{-2n}$$

if $p, g(p), \ldots, g^n(p) \notin B(0, r_1)$. On the other hand, by Lemma 5.1, as long as the angle $\eta(n) \geq \tilde{\rho}$, its value doesn't increase as p passes through $B(0, r_1)$. If forward trajectories of all the points in γ are infinitely often outside of $B(0, r_1)$, then for some $n \geq 0$, the curve $g^n(\gamma)$ is fully contained in an unstable cone and this case has been considered above. Otherwise consider a point $p \in \gamma$ and $n_0 > 0$ such that $g^n(p) \in B(0, r_1)$ for all $n \geq n_0$. We then must have that $g^n(p) \in W^s_{0,f}$. Since the angle between $T_{g^n(p)}\gamma$ and the stable subspace of $g^n(p)$ (for f) is bounded away from zero, the endpoints of γ eventually exit Z. In particular, for some $m \geq n$ we get that $d(g^m(x), g^m(y)) > \bar{\delta}$.

This proves the expansiveness of g. Note in addition that the expansivity constant can only grow as we decrease r_1. Indeed decreasing r_1 does not change the estimates (3.3)-(3.6), but it increases the value of Q in Remark 5.1 and the value of $\bar{\delta}$. □

Proof of Statement 2 in Theorem 3.1. The proof uses the following result (compare with Theorem 18.1.3 in [6] and Theorem 1.4.1 in [5]).

Lemma 5.2 (Anosov Shadowing Theorem). *If M is a Riemannian manifold, $U \subset M$ open, $f : U \to M$ a C^1 embedding, then any compact, locally maximal, hyperbolic set $\Lambda \subset U$ for f has a neighborhood V and $\epsilon_0, \delta_0, C > 0$ such that if $f' : V \to M$, $d_{C^1}(f, f') < \epsilon_0$, $\Lambda' = \bigcap_{n\in\mathbb{Z}} f'^n\bar{V}$, Y is a topological space, $\sigma : Y \to Y$ a homeomorphism, $\alpha \in C^0(Y, V)$, and $d_{C^0}(\alpha\sigma, f'\alpha) := \sup_{y\in Y} d(\alpha\sigma(y), f'\alpha(y)) < \epsilon < \epsilon_0$, then there is an $h \in C^0(Y, \Lambda')$ with $h\sigma = f'h$ and $d_{C^0}(\alpha, h) < C\epsilon$. Moreover, h is locally unique: $h'\sigma = f'h'$ and $d_{C^0}(\alpha, h') < \delta_0$ implies $h' = h$.*

In the above theorem take $f = f'$ to be the unperturbed map, $Y = \Lambda_g$, and $\sigma = g_{|\Lambda_g}$. Let V be as in the theorem. Note that taking r_1 sufficiently small we obtain that $\Lambda_g \subset V$ (it is enough to ensure that $B(0, r_1) \subset V$). Moreover, the set Λ_g is g-invariant and the perturbation is local, i.e., f and g are C^0 close in U. Therefore if $r_1 > 0$ is small enough, then for any point $x \in \Lambda_g$, its image $f(x) \in V$. Let then $\alpha = f \circ g^{-1}$. We have that $\alpha(\Lambda_g) \subset V$ as assumed in the theorem. We check that,

$$d_{C^0}(Id_{|\Lambda_g}, \alpha) = d_{C^0}(Id_{|\Lambda_g}, f \circ g^{-1}_{|\Lambda_g}) = d_{C^0}(f \circ f^{-1}_{|\Lambda_g}, f \circ g^{-1}_{|\Lambda_g}) \leq Ld_{C^0}(f^{-1}_{|\Lambda_g}, g^{-1}_{|\Lambda_g}), \tag{5.11}$$

where L is the Lipschitz constant for f. Then,

$$d_{C^0}(\alpha\sigma, f'\alpha) = d_{C^0}(f \circ g^{-1}g_{|\Lambda_g}, f \circ f \circ g^{-1}_{|\Lambda_g}) = d_{C^0}(f_{|\Lambda_g}, f \circ \alpha) \leq Ld_{C^0}(Id_{|\Lambda_g}, \alpha) \leq L^2 d_{C^0}(f^{-1}_{|\Lambda_g}, g^{-1}_{|\Lambda_g}).$$

Note that by taking $r_1 > 0$ sufficiently small we can guarantee that the distance $d_{C^0}(f^{-1}_{|\Lambda_g}, g^{-1}_{|\Lambda_g})$ is small enough, so that $d_{C^0}(\alpha\sigma, f'\alpha) < \epsilon < \epsilon_0$ as in the theorem. Therefore there exists a continuous map $h : \Lambda_g \to \Lambda$ such that,

$$d_{C^0}(\alpha, h) \leq CL^2 d_{C^0}(f^{-1}_{|\Lambda_g}, g^{-1}_{|\Lambda_g}) < C\epsilon \tag{5.12}$$

and

$$h \circ g_{|\Lambda_g} = f \circ h. \tag{5.13}$$

By (5.11) and (5.12),

$$d_{C^0}(Id_{|\Lambda_g}, h) \leq d_{C^0}(Id_{|\Lambda_g}, \alpha) + d_{C^0}(\alpha, h) \leq Ld_{C^0}(f^{-1}_{|\Lambda_g}, g^{-1}_{|\Lambda_g}) + CL^2 d_{C^0}(f^{-1}_{|\Lambda_g}, g^{-1}_{|\Lambda_g}) < \epsilon(\frac{1}{L} + C). \tag{5.14}$$

To prove the conjugacy it is then enough to show that h is invertible. Indeed, taking any closed set $D \subset \Lambda$ we can see that D is in fact compact. Then if h is invertible, the preimage under h^{-1} of D is the set $(h^{-1})^{-1}(D) = h(D)$. Since h is continuous, $h(D)$ is compact. That shows that if h is invertible, then h^{-1} is continuous.

Injectivity of h follows from the fact that we can choose r_1 sufficiently small, so that the expansivity constant of $g_{|\Lambda_g}$ can be estimated from below as $\delta \geq 2(\frac{1}{L} + C)L^2 d_{C^0}(f^{-1}_{|\Lambda_g}, g^{-1}_{|\Lambda_g})$. To see this assume that h is not "$1-1$". Then there are $x, y \in \Lambda_g$ with $x \neq y$ and such that $h(x) = h(y)$. Then also $f(h(x)) = f(h(y))$. Using (5.13) gives that $h(g(x)) = h(g(y))$. Taking $\tilde{x} = g(x)$ and $\tilde{y} = g(y)$ and repeating these reasoning gives that $h(g^2(x)) = h(g^2(y))$. Continuing in this manner we get $h(g^n(x)) = h(g^n(y))$ for any positive integer n. On the other hand, if $h(g^{-1}(x)) \neq h(g^{-1}(y))$, the equation (5.13) and injectivity of f give

$$h(x) = h(g(g^{-1}(x))) = f(h(g^{-1}(x))) \neq f(h(g^{-1}(y))) = h(g(g^{-1}(y))) = h(y).$$

We then see that if $h(x) = h(y)$, then necessarily $h(g^n(x)) = h(g^n(y))$ for all $n \in \mathbb{Z}$.

By (5.14) we have that $d(g^n(x), g^n(y)) \leq 2(\frac{1}{L} + C)L^2 d_{C^0}(f^{-1}_{|\Lambda_g}, g^{-1}_{|\Lambda_g})$ for all $n \in \mathbb{Z}$. If now g is expansive with expansivity constant no less than $2(\frac{1}{L} + C)L^2 d_{C^0}(f^{-1}_{|\Lambda_g}, g^{-1}_{|\Lambda_g})$, we get a contradiction.

We now show that h is surjective. First observe that the image $h(\Lambda_g) =: \Lambda'$ is $f-$invariant. Indeed,

$$\Lambda' = h(\Lambda_g) = h(g(\Lambda_g)) = f(h(\Lambda_g)) = f(\Lambda').$$

Note also that Λ_g contains $W^u_{0,f}$, a local unstable manifold of zero for f. Indeed, since zero is a fixed point and the trajectories of the flows generating f and g are identical, we have that

$$W^u_{0,f} := \bigcap_{n\geq 0} \overline{f^n(B(0,r_1))} = \bigcap_{n\geq 0} \overline{g^n(B(0,r_1))}.$$

By (5.13), we have that $h(0) = h(g(0)) = f(h(0))$ is a fixed point for f. In addition, (5.14) gives that $h(0)$ is close to 0. Since the origin is a hyperbolic fixed point for f, it is isolated and hence, taking r_1 sufficiently small we ensure that $h(0) = 0$. Note also that taking any backward trajectory $\{g^{-n}(x)\}_{n\geq 0} \subset W^u_{0,f}$, by (5.13), $h(\{g^{-n}(x)\}_{n\geq 0}) = \{f^{-n}(h(x))\}_{n\geq 0}$ and by continuity of h we have that $\lim_{n\to\infty} f^{-n}(h(x)) = \lim_{n\to\infty} h(g^{-n}(x)) = h(0) = 0$. We then conclude that $\tilde{W} := h(W^u_{0,f})$ is itself a local unstable manifold of 0 for f.

Since Λ' is f-invariant, it implies that

$$\bigcup_{n\geq 0} f^n(\tilde{W}) \subset \Lambda'.$$

On the other hand, $\bigcup_{n\geq 0} f^n(\tilde{W})$ is exactly the global unstable manifold of 0 (for f) and as such is dense in Λ. Because h is continuous and Λ_g is compact, it also gives that,

$$\Lambda = \overline{\bigcup_{n\geq 0} f^n(\tilde{W})} \subset \Lambda' \subset \Lambda.$$

□

Proof of Statements 1 and 2 in Theorem 3.3. We first prove that $E^u_g(x)$ is a one-dimensional subspace of T_xM. Our goal is to show that for any two vectors $v_n, w_n \in K^u_\rho(g^{-n}(x))$, the positive angle $\eta(n) := \angle(Dg^n(v_n), Dg^n(w_n))$ tends to zero as n tends to infinity.

We start with the case when the backward semi-trajectory $g^{-n}(x)$ is infinitely often outside $B(0,r_1)$. Then it suffices to show that taking a point $\tilde{x} := g^{-\mathcal{N}}(x)$ and considering two vectors $v, w \in K^u_\rho(\tilde{x})$, the positive angle $\tilde{\eta}(n) := \angle(Dg^n(v), Dg^n(w))$ (where $n \le \mathcal{N}$) is arbitrarily small for n and $\mathcal{N}$ large enough.

We first study how $\tilde{\eta}$ changes as $g^n(\tilde{x})$ passes through the perturbed region. To this end consider a point $x' \in g^{-1}(B(0,r_1)) \setminus B(0,r_1)$ and $v', w' \in K^u_\rho(x')$. Denote $\eta' := \angle(v', w')$ to be a positive acute angle. Assume that x' exits $B(0,r_1)$ at time N and denote $\bar{x}' := g^N(x')$, $\bar{v}' := Dg^N(v')$, $\bar{w}' := Dg^N(w')$, and $\bar{\eta}' = \angle(\bar{v}', \bar{w}')$. Without loss of generality we may assume that $\|\bar{v}'\| = \|\bar{w}'\| = 1$. Note that then $(\bar{w}' - \bar{v}') \in K^s_\rho(\bar{x}')$. In addition the vectors $\bar{v}', \bar{w}'$ and $\bar{w}' - \bar{v}'$ can be considered as sides of an isosceles triangle on a plane. The line l passing through $\bar{x}'$ and perpendicular to $\bar{w}' - \bar{v}'$ intersects $\bar{w}' - \bar{v}'$ exactly in the middle and divides $\bar{\eta}'$ in half. We then have

$$\sin\frac{\bar{\eta}'}{2} = \frac{1}{2}\|\bar{w}' - \bar{v}'\|. \tag{5.15}$$

By (3.4) in Proposition 3.2, $\|\bar{v}'\| \ge C\|v'\|$. On the other hand, by (3.6) in Proposition 3.2, $\|\bar{w}' - \bar{v}'\| \le \frac{1}{C}\|w' - v'\|$. Then (5.15) gives that

$$\sin\frac{\bar{\eta}'}{2} \le \frac{\frac{1}{2C^2}\|w' - v'\|}{\|v'\|}. \tag{5.16}$$

Note that since $\eta' < \frac{\pi}{2}$, then $\sin\eta' \ge \sin\eta_P$, where η_P is the angle between v' and the vector $\mathbf{n} \in Span(v', w')$ orthogonal to $v' - w'$. Therefore, we may assume that

$$\sin\eta' \ge \frac{\frac{1}{2}\|w' - v'\|}{\|v'\|}.$$

If this is not the case, we shall consider w' instead of v' in the above inequality as well as in (5.15) and (5.16). We then obtain, $\sin\eta' \ge C^2 \sin\frac{\bar{\eta}'}{2}$. Consequently,

$$\sin\bar{\eta}' = 2\sin\frac{\bar{\eta}'}{2}\cos\frac{\bar{\eta}'}{2} \le \frac{2}{C^2}\sin\eta'. \tag{5.17}$$

We now need similar estimates for a piece of orbit which lies outside $B(0,r_1)$. Suppose $x'', g(x''), \ldots, g^N(x'') \in \Lambda_g \setminus B(0,r_1)$. As before, consider $v'', w'' \in K^u_\rho(x'')$ and the positive angle $\eta'' := \angle(v'', w'')$. Denote $\bar{x}'' := g^N(x'')$, $\bar{v}'' := Dg^N(v'')$, $\bar{w}'' := Dg^N(w'')$, and $\bar{\eta}'' = \angle(\bar{v}'', \bar{w}'')$. Without loss of generality we may assume that $\|\bar{v}''\| = \|\bar{w}''\| = 1$. Then as in (5.15) we have that

$$\sin\frac{\bar{\eta}''}{2} = \frac{1}{2}\|\bar{w}'' - \bar{v}''\|. \tag{5.18}$$

By (3.3) in Proposition 3.2, $\|\bar{v}''\| \ge C\nu^N\|v''\|$, while by (3.5) in Proposition 3.2, $\|\bar{w}'' - \bar{v}''\| \le C\nu^{-N}\|w'' - v''\|$. Consequently, we obtain that

$$\sin\bar{\eta}'' \le \tilde{C}\nu^{-2N}\sin\eta'' \tag{5.19}$$

for some $\tilde{C} > 0$ not depending on x'' and the tangent vectors.

By Remark 5.1, for any $Q > 0$ we can choose r_1 small enough so that every point exiting $B(0,r_1)$ spends at least Q steps in the complement of $B(0,r_1)$ before returning to $B(0,r_1)$. We can choose Q large enough to ensure that $\frac{2}{C^2}\tilde{C}\nu^{-2Q} < 1$. Then, by (5.17) and (5.19), we have that $\sin\tilde{\eta}(n)$ contracts as $(\frac{2}{C^2}\tilde{C}\nu^{-2Q})^k$, where k counts the number of exits of $\tilde{x}$ from $B(0,r_1)$. In particular $\tilde{\eta}(n)$ is arbitrarily small for sufficiently large n.

We now turn to the case when $g^{-n}(x)$ spends finite time outside $B(0, r_1)$. This means that some backward iterate, $g^{-n}(x) \in B(0, r_1) \cap W^u_{0,f}$, where $W^u_{0,f}$ denotes a local unstable manifold of 0 for the map f. Then it is enough to consider $x \in B(0, r_1) \cap W^u_{0,f}$. Recall that in local coordinates at 0 the map g is the time one map of the flow φ_t generated by the vector field $\chi(x) = \psi(x)Ax$, where $A = \gamma Id_u - \beta Id_s$ for some $\beta > \gamma > 0$ (see Condition (C4) and (D0)). Set $x_0 \in W^u_{0,f}$ arbitrarily close to zero. Consider $\{x(t) \mid 0 \le t \le T\}$, a piece of trajectory of the flow φ_t and such that $x(0) = x_0, x(T) = x$. We claim that taking any $v \in K^u_\rho(x_0)$, the angle between $v_T := D\varphi_T(x(0))(v)$ and $E^u_f(x)$ is arbitrarily close to zero if x_0 is sufficiently close to the origin. From this we conclude that $E^u_g(x) = E^u_f(x)$ for $x \in B(0, r_1)$, then also $E^u_g(g^n(x)) = Dg^n(E^u(x))$.

Denote $v(t) := D\varphi_t(x(0))(v)$. Let $\rho(t)$ be the positive angle between $v(t)$ and $E^u_f(x(t))$. We write $v = v_u + v_s$ where v_u is the projection on E^u_f and v_s is the projection on E^s_f. Assume that both v_u and v_s are nonzero. Note that $x(t) = x_u(t)$. By Proposition 7.6 in [3] we obtain,

$$\|v_u\|' = \frac{1}{\|v_u\|} \left\langle v_u, \alpha\|x\|^{\alpha-2}\|v_u\|\|x_u\|Ax_u + \|x\|^\alpha Av_u \right\rangle = \|x\|^\alpha\|v_u\|\gamma(\alpha+1), \text{ while}$$

$$\|v_s\|' = \frac{1}{\|v_s\|} \langle v_s, \|x\|^\alpha Av_s \rangle = -\beta\|v_s\|\|x\|^\alpha.$$

Consequently,

$$(\tan\rho)' = \left(\frac{\|v_s\|}{\|v_u\|}\right)' = \|x\|^\alpha \frac{\|v_s\|}{\|v_u\|}(-\beta - \gamma(\alpha+1)) = -\|x\|^\alpha(\beta + \gamma(\alpha+1))\tan\rho. \tag{5.20}$$

On the other hand,

$$\left(\|x\|^{-\alpha}\right)' = -\alpha\left\langle \frac{x}{\|x\|}, \frac{Ax}{\|x\|} \right\rangle = -\alpha\left\langle \frac{x}{\|x\|}, \frac{\gamma x}{\|x\|} \right\rangle = -\alpha\gamma.$$

Solving this differential equation yields,

$$\|x(t)\|^\alpha = \frac{1}{-\alpha\gamma t + \|x(0)\|^{-\alpha}}.$$

Substituting in (5.20) gives,

$$(\tan\rho)' = -\frac{1}{-\alpha\gamma t + \|x(0)\|^{-\alpha}}(\beta + \gamma(\alpha+1))\tan\rho.$$

Finally, solving the above differential equation for $\tan\rho$ gives,

$$\tan\rho(t) = \tan\rho(0)\left(\frac{\|x(0)\|}{\|x(t)\|}\right)^{\frac{\beta+\gamma(\alpha+1)}{\gamma}},$$

so that

$$\tan\rho(T) = \tan\rho(0)\left(\frac{\|x_0\|}{\|x\|}\right)^{\frac{\beta+\gamma(\alpha+1)}{\gamma}}.$$

The above quantity tends to zero as $\|x_0\| \to 0$.

We now show that $E^s_g(x)$ is a $(dimM - 1)$-dimensional subspace. We start with the case when $g^n(x)$ is infinitely often outside $B(0, r_1)$. Then it suffices to show that taking a point $g^n(x) =: \tilde{x} \in \Lambda_g$ and considering two vectors $v, w \in K^s_\rho(\tilde{x})$ for which $(v - w) \in E^u_g(\tilde{x})$, the positive angle $\eta(n) := \angle(Dg^{-n}(v), Dg^{-n}(w))$ tends to zero as n tends to infinity.

Since $E^u_g(x)$ is Dg-invariant, we obtain that

$$(Dg^{-n}(v) - Dg^{-n}(w)) = Dg^{-n}(v - w) \in E^u_g(x)$$

for all $n \geq 0$. Then we can estimate

$$\sin \eta(n) \leq \frac{\|Dg^{-n}(v-w)\|}{\|Dg^{-n}(v)\|}.$$

By (3.3) and (3.4) in Proposition 3.2,

$$\|Dg^{-n}(v-w)\| \leq \left(\frac{\nu^{-Q}}{C^2}\right)^k \|v-w\|,$$

where $k = k(n)$ counts the number of exits of the trajectory of x of length n from $B(0, r_1)$ and $\frac{\nu^{-Q}}{C^2} < 1$. Similarly, by (3.5) and (3.6) in Proposition 3.2,

$$\|Dg^{-n}(v)\| \geq \left(\nu^Q C^2\right)^k \|v\|,$$

where k counts the number of exits of x from $B(0, r_1)$ and $\nu^Q C^2 > 1$. Together we have for some $0 < \tilde{C} < 1$ that

$$\sin \eta(n) \leq \tilde{C}^k \frac{\|v-w\|}{\|v\|} \to 0 \text{ as } n \to \infty.$$

We now turn to the case when $g^n(x)$ spends finite time outside of $B(0, r_1)$. This means that some iterate, $g^n(x) \in B(0, r_1) \cap W^s_{0,f}$, where $W^s_{0,f}$ denotes a local stable manifold of 0 for the map f. Then it is enough to consider $x \in B(0, r_1) \cap W^s_{0,f}$. Set $x_0 \in W^s_{0,f}$ arbitrarily close to zero. Consider $\{x(-t) \mid 0 \leq t \leq T\}$, a piece of backward trajectory of the flow φ_t and such that $x(0) = x_0, x(-T) = x$. We claim that taking any $v \in K^s_\rho(x_0)$, the angle between $v_T := D\varphi_{-T}(x(0))(v)$ and $E^u_f(x)$ is arbitrarily close to zero if x_0 is sufficiently close to the origin. From this we conclude that $E^s_g(x) = E^s_f(x)$ for $x \in B(0, r_1)$.

Denote $v(t) := D\varphi_{-t}(x(0))(v)$. Let $\rho(t)$ be the positive angle between $v(t)$ and $E^u_f(x(-t))$. We write $v = v_u + v_s$ where v_u is the projection on E^u_f and v_s is the projection on E^s_f. Note that $x(-t) = x_s(-t)$. Assuming that v_u and v_s are both nonzero, by Proposition 7.6 in [3] we derive,

$$\begin{aligned}
\|v_u\|' &= \frac{-1}{\|v_u\|} \langle v_u, \|x\|^\alpha A v_u \rangle = -\|x\|^\alpha \|v_u\| \gamma, \text{ while} \\
\|v_s\|' &= \frac{-1}{\|v_s\|} \left\langle v_s, \alpha \|x\|^{\alpha-2} \langle v, x \rangle A x_s + \|x\|^\alpha A v_s \right\rangle = q_s \beta \alpha \|x\|^{\alpha-2} + \|x\|^\alpha \beta \|v_s\|,
\end{aligned}$$

where $q_s := \frac{\langle v_s, x_s \rangle^2}{\|v_s\|} \geq 0$. Consequently,

$$(\tan \rho)' = \left(\frac{\|v_u\|}{\|v_s\|}\right)' \leq -\frac{\|x\|^\alpha \|v_u\|}{\|v_s\|}(\alpha + \beta) = -\|x\|^\alpha (\alpha + \beta) \tan \rho. \tag{5.21}$$

On the other hand,

$$\left(\|x(-t)\|^{-\alpha}\right)' = \alpha \left\langle \frac{x}{\|x\|}, \frac{Ax}{\|x\|} \right\rangle = -\alpha\beta.$$

Solving above differential equation gives,

$$\|x(-t)\|^\alpha = \frac{1}{-\alpha\beta t + \|x(0)\|^{-\alpha}}.$$

Substituting in (5.21) and solving for $\tan \rho$ gives,

$$\tan \rho(t) \leq \tan \rho(0) \left(\frac{\|x(0)\|}{\|x(-t)\|}\right)^{\frac{\beta+\gamma}{\beta}},$$

so that

$$\tan \rho(T) \leq \tan \rho(0) \left(\frac{\|x_0\|}{\|x\|} \right)^{\frac{\beta+\gamma}{\beta}},$$

The above quantity tends to zero as $\|x_0\| \to 0$. □

Proof of Statement 3 in Theorem 3.3. Take a sequence $\{x_n\} \subset \Lambda_g$ such that $x_n \to x \in \Lambda_g$. Assume there exists a sequence of tangent vectors $v_n \in E^u_g(x_n)$ such that $v_n \to v$ and $\angle(v, E^u_g(x)) = \delta > 0$. By Statement 1, there exists N such that $Dg^N K^u_\rho(g^{-N}(x)) \subset K(x, E^u_g(x), \delta/2)$, so that $v \notin Dg^N K^u_\rho(g^{-N}(x))$. On the other hand, there exists a sequence of tangent vectors $\tilde{v}_n \in K^u_\rho(g^{-N}(x_n))$ such that $Dg^N(g^{-N}(x_n))(\tilde{v}_n) = v_n$. By continuity of Dg we have that $\tilde{v}_n \to \tilde{v}$, where $Dg^N(g^{-N}(x))(\tilde{v}) = v$. Since the family K^u_ρ is continuous, we have that $\tilde{v} \in K^u_\rho(g^{-N}(x))$. This implies that $v \in K(x, E^u_g(x), \delta/2)$ leading to a contradiction. Similarly, continuity of the stable cone family and Statement 2 implies continuity of $E^s_g(x)$. □

Proof of Statement 4 in Theorem 3.3. Recall that in the proof of Statements 1 and 2 we showed that $E^u_g(x) = E^u_f(x) = T_x W^u_{0,f}$ for $x \in W^u_{0,f}$, and that $E^s_g(x) = E^s_f(x) = T_x W^s_{0,f}$ for $x \in W^s_{0,f}$. Therefore the submanifolds $V^u_0 = \bigcup_{n \geq 0} g^n(W^u_{0,f})$ and $V^s_0 = \bigcup_{n \geq 0} g^{-n}(W^s_{0,f})$ satisfy the assertion of Statement 4 at every point.

We now prove the existence and regularity of W^s_x for $x \notin V^s_0$. The existence and regularity of W^u_x for $x \notin V^u_0$ is proved using the same argument with the reversed time. The proof uses the following lemma.

Lemma 5.3. *For any $\epsilon > 0$ there is $\kappa > 0$ such that for any $x \in \Lambda_g$ and for any $(\dim M - 1)$-dimensional submanifolds $W, \tilde{W} \subset \overline{B(x, \kappa)}$ intersecting at x and such that $TW \subset K^s_\rho$ and $T\tilde{W} \subset K^s_\rho$, the following is true. If ζ is a curve with endpoints at W and $\tilde{W}$ respectively, and $T\zeta \subset K^u_\rho$, then the length of ζ doesn't exceed ϵ.*

Proof of Lemma 5.3. In local coordinates we identify x with 0, $E^s_f(x)$ with $\mathbb{R}^{d-1}$ and $E^u_f(x)$ with $\mathbb{R}$. Note that since the angle between E^u_f and E^s_f is bounded uniformly away from zero, the distortion produced by such an identification is uniformly bounded. Since the cone family $K^s_\rho(x)$ is continuous, we can pick κ small enough so that $W, \tilde{W}$ are contained in the set $S := \{(x, y) \in \mathbb{R}^{d-1} \times \mathbb{R} \mid \|x\| < \xi \|y\|\}$ for some $0 < \xi < 1$. We also have that $\|\zeta(0)\|, \|\zeta(1)\| < \kappa$. By continuity of the cone family $K^u_\rho(x)$, if κ is small enough, ζ is contained in the set $\tilde{S} := \{(x, y) \in \mathbb{R}^{d-1} \times \mathbb{R} \mid \|y - \zeta(0)_u\| < \xi \|x - \zeta(0)_s\|\}$. One can see that the upper bound for the length of ζ goes to 0 as κ goes to zero. The fact that κ doesn't depend on x follows from compactness of Λ_g. □

On Λ_g fix a family $W(x)$ of $C^{1+\alpha}$ embedded discs centered at x and such that $TW(x) \subset K^s_\rho$ for every $x \in \Lambda_g$ (see Remark 3.1). Fix $x' \in \Lambda_g$. Let W_n denote the connected component of $g^{-n}(W(g^n(x'))) \cap \overline{B(x', \kappa)}$ containing x'. Here we picked κ sufficiently small, so that the corresponding ϵ from Lemma 5.3 is smaller than the expansivity constant for g. Then $\{W_n\}$ is a sequence of $C^{1+\alpha}$ embedded discs centered at x'. We first show that $\{W_n\}$ is a Cauchy sequence in the C^0 topology. Take large m and n with $n > m$. Note that by the invariance of the cone family, $K^s_\rho(x)$, we have that $TW_m, TW_n \subset K^s_\rho$. Hence if $W_m \neq W_n$, we can find a curve ζ with endpoints at W_m and W_n respectively and such that $T\zeta \subset K^u_\rho$. Let the length of the longest of all such curves be $\epsilon_1 > 0$. Then $g^m(\zeta)$ is a curve connecting $W(g^m(x'))$ and $g^{m-n}(W(g^n(x')))$. By the invariance of the cone family $K^u_\rho(x)$, we have that $Tg^m(\zeta) \subset K^u_\rho$, while $TW(g^m(x')), Tg^{m-n}(W(g^n(x'))) \subset K^s_\rho$. By Lemma 5.3, the length of $g^m(\zeta)$ cannot exceed ϵ, which is smaller than the expansivity constant. On the other hand, the length of $g^i(\zeta)$ expands proportionally to $(C^2\nu^Q)^{k(i)}$, where C is a constant from Proposition 3.2 and $k(i)$ counts the number of exits of x from $B(0,r_1)$. If $x' \notin V_0^s$, then $k(i) \to \infty$ as $i \to \infty$. This shows that W_n is a Cauchy sequence in the C^0 topology.

We now show that the convergence holds in C^1. Consider a sequence of points $p_m \in W_m$ such that $p_m \to p$. Fix a large M and let $n > m$ be large enough so that $\|Dg^M_{p_m} - Dg^M_{p_n}\| < \epsilon_0$ for some small $\epsilon_0 > 0$. Consider unit vectors $v_m \in T_{p_m}W_m, v_n \in T_{p_n}W_n$. Assume that $\|v_m - v_n\| > \epsilon_1$ for some small $\epsilon_1 > 0$. Then we have that

$$\|Dg^M_{p_m}(v_m) - Dg^M_{p_n}(v_n)\| \geq \|Dg^M_{p_m}(v_m - v_n)\| - \epsilon_0 \geq (C^2\nu^Q)^{k(M)}\epsilon_1 - \epsilon_0.$$

On the other hand,

$$\|Dg^M_{p_m}(v_m) - Dg^M_{p_n}(v_n)\| \leq \|Dg^M_{p_m}(v_m)\| + \|Dg^M_{p_n}(v_n)\| \leq 2\left(\frac{1}{C^2\nu^Q}\right)^{k(M)},$$

where C is a constant from Proposition 3.2 and $k(M)$ denotes the number of exits of x' from $B(0,r_1)$. Together we obtain that

$$(C^2\nu^Q)^{k(M)}\epsilon_1 - \epsilon_0 \leq 2\left(\frac{1}{C^2\nu^Q}\right)^{k(M)}.$$

We can see that the value of M for which the above inequality holds is bounded uniformly in p. □

5.2. **Proof of Theorem 4.2.**

Proof of Theorem 4.2. Consider a Markov partition $\mathcal{R}$ for f. Using the conjugacy map h, we obtain a Markov partition $\mathcal{P} := h^{-1}(\mathcal{R})$ for g. Given $Q > 0$ we can choose $r_1 > 0$ small enough so that if $\mathcal{P}$ has sufficiently small diameter, then there exists $P \in \mathcal{P}$ with the following properties. Every piece of trajectory of g that starts from P and ends in $B(0,r_1)$ has length at least Q; and every piece of trajectory of g that starts from $B(0,r_1)$ and ends in P has length at least Q. To ensure that it is enough to take a small $r_1 > 0$ and $P \in \mathcal{P}$ which does not intersect with Z (see Condition $(C4)$ for the definition of Z). Recall that by the Remark 5.1, we also have that every piece of trajectory of g that exits and returns back to $B(0,r_1)$ has length at least Q.

The set P has a hyperbolic product structure. Closed segments of local unstable manifolds, $W^u_x \cap P$ for $x \in P$, satisfy the definition of a continuous family of unstable discs introduced in Section 4. Indeed, taking $x, y \in W^u_x \cap P$ we have that $d(f^{-n}(h(x)), f^{-n}(h(y))) \to 0$ as $n \to \infty$. This implies that $d(g^{-n}(x), g^{-n}(y)) \to 0$ as $n \to \infty$. To obtain a continuous family of stable discs, for every $x \in P$ consider the connected component of $W^s_x \setminus \partial^u P$ containing x, where $\partial^u P$ denotes the unstable boundary of P (see for example Definition 18.7.1 in [6]). This proves $(A0)$. To prove Condition $(A1)$ consider $\tau(x)$, the first return time to P. Then for every $n \geq 0$ we assign a finite collection, $\{P^s_{i_n}\}$, of connected components of the set $\{x \in P | \tau(x) = n\}$. Then $(A1)$ follows from the fact that P is an element of a Markov partition. To show $(A2)(a)$, note that outside of $B(0,r_1)$ stable curves are uniformly contracted. Then, if Q is large enough, $(A2)(a)$ follows by the inequality (5.30), which is presented later. Condition $(A2)(b)$ can be proven similarly, using the inequality (7.43) in [3]. To prove $(A5)$ note that the inducing time is the first return time. We can then use Kac's formula with respect an SRB-measure μ_1 for g whose existence is proved in ([3], Theorem 2.4). We then obtain $\sum_{i=1}^{\infty} \tau_i \mu_1(P^s_i) < \infty$. This remains true if we substitute μ_1 by the conditional measure $\mu_{1_{\gamma^u}}$. But $\mu_{1_{\gamma^u}}$ is absolutely continuous with respect to m_{γ^u}. The first statement of Condition $(A4)$ holds since

$x \in (P \setminus \cup P_i^s \cap \gamma^u)$ implies that either x lies on the boundary of the Markov partition or it never returns to $IntP$. The same is true for $x \in (scl(P_i^s) \setminus P_i^s)$. From this we also see that the set $\bigcup_{i \in \mathbb{N}} (scl(P_i^s) \setminus P_i^s)$ cannot support any measure, which gives positive weight to any set open in Λ thus proving $(A6)$. The second statement of Condition $(A4)$ follows by noticing that since Λ_g is an attractor, the unstable manifolds are entirely contained in Λ_g, so that $m_{W_x^u}(W_x^u \cap P) > 0$ for every $x \in P$.

We now turn to the proof of $(A3)$. We will show $(A3)(a)$. The essential part of the proof is to find a uniform bound on distortion along stable discs in the perturbed area. $(A3)(b)$ can be then proved similarly, using estimates for distortion along unstable discs in the perturbed area presented in ([3], Section 7.4).

To simplify notation from now on we will use the letter C to denote a universal positive constant instead of enumerating different constants.

First we are going to show a series of technical lemmas describing behavior of trajectories in $Y := B(0, r_1)$. Let $x : [0, T] \to Y$ be the trajectory of the flow generated by $\chi := \|x\|^\alpha Ax$. For a fixed $t \in [0, T]$ let $\theta(t)$ be the positive angle between the vector $\vec{0x}$ and $E^u(0)$. Let T be the time at which x leaves Y so that $\|x(0)\| = \|x(T)\| = r_1$ and $x(t) \in Y$ for all $t \in [0, T]$. Denote $\lambda = \gamma + \beta$. Define

$$J(t_1, t_2) := \lambda \int_{t_1}^{t_2} \|x(\tau)\|^\alpha d\tau.$$

The following proposition describes relations between the angle θ and the norm of a given point x.

Proposition 5.1. *([3], formula (7.2) and Lemma 7.2)*

(1) For all $0 < t_1 < t_2 < T$ we have that

$$\tan\theta(t_2) = e^{-J(t_1, t_2)} \tan\theta(t_1);$$

(2) for all $0 < t < T$ we have that

$$\frac{d}{dt}\|x(t)\|^{-\alpha} = \alpha \frac{\beta \tan^2\theta(t) - \gamma}{\tan^2\theta(t) + 1}.$$

Let T_1 denote the time for which $\tan\theta = 1$. The following lemma controls how $\|x\|$ changes.

Lemma 5.4. *For all $t \in [0, T]$ we have,*

$$\|x(t)\|^\alpha \geq r_1^\alpha (1 + r_1^\alpha \alpha\beta t)^{-1} \geq C(t+1)^{-1}, \text{ equivalently,} \tag{5.22}$$

$$\|x(t)\|^{\alpha - 1} \leq C(t+1)^{-1+\frac{1}{\alpha}}. \tag{5.23}$$

In addition, for all $t \in [0, T_1]$,

$$\|x(t)\|^\alpha \leq C(t+1)^{-1}. \tag{5.24}$$

Proof. By Statement 2 in Proposition 5.1 we have that $\frac{d}{dt}\|x\|^{-\alpha} \leq \alpha\beta$. Consequently,

$$\|x(t)\|^{-\alpha} - \|x(0)\|^{-\alpha} \leq \alpha\beta t, \text{ so that}$$
$$\|x(t)\|^{-\alpha} \leq r_1^{-\alpha} + \alpha\beta t \leq C(1+t).$$

This shows (5.22) and (5.23). To prove (5.24) observe that by Statement 1 in Proposition 5.1 we have that $\tan\theta(t) \geq 1$ for all $t \leq T_1$. Note also, that for $\beta > \gamma$ (see Condition $(C4)$), the function

$\psi(s) := \frac{\beta s^2-\gamma}{s^2+1}$ increases on the interval $(0,\infty)$. Then Statement 2 in Proposition 5.1 gives that for all $t \le T_1$,

$$\begin{aligned}
&\frac{d}{dt}\|x\|^{-\alpha} \ge \alpha\frac{\beta-\gamma}{2} := \xi > 0. \text{ Consequently,}\\
&\|x(t)\|^{-\alpha} - \|x(0)\|^{-\alpha} \ge \xi t, \text{ and}\\
&\|x(t)\|^{-\alpha} \ge r_1^{-\alpha} + \xi t \ge C(1+t).
\end{aligned}$$

□

Using the above lemma we now estimate $\int \|x\|^\alpha$.

Lemma 5.5. *For any* $0 < t_1 < t_2 < T$ *we have that*

$$\int_{t_1}^{t_2} \|x\|^\alpha dt \ge -C + \frac{1}{\alpha\beta}\log\left(\frac{t_2+1}{t_1+1}\right). \tag{5.25}$$

Proof. Using Lemma 5.4 we can write,

$$\begin{aligned}
\int_{t_1}^{t_2} \|x\|^\alpha dt &\ge \int_{t_1}^{t_2} r_1^\alpha(1+r_1^\alpha\alpha\beta t)^{-1} = \frac{1}{\alpha\beta}\log(1+r_1^\alpha\alpha\beta t)|_{t_1}^{t_2}\\
&= \frac{1}{\alpha\beta}\log\left(\frac{1+r_1^\alpha\alpha\beta t_2}{1+r_1^\alpha\alpha\beta t_1}\right) \ge -C + \frac{1}{\alpha\beta}\log\left(\frac{t_2+1}{t_1+1}\right).
\end{aligned}$$

□

We now need to study behavior of tangent vectors in Y. For a vector $v \in T_xM$ we will write $v = v_s + v_u$ with $v_s \in E^s(0)$ and $v_u \in E^u(0)$. Let ρ_u be the positive angle between v and E^u and let ρ_s be the positive angle between v and E^s, so that $\tan\rho_u = \frac{\|v_s\|}{\|v_u\|}$ and $\tan\rho_s = \frac{\|v_u\|}{\|v_s\|}$. Denote $\hat{x} := \frac{x}{\|x\|}$. Recall that T_1 is such that $\tan\theta(T_1) = 1$.

The following proposition describes the behavior of tangent vectors.

Proposition 5.2. *([3], formula (7.21), Lemma 7.8 and Corollary 7.9) For all* $0 < t < T$ *we have that*

(1) $\dot{v}(t) = \|x\|^\alpha(\alpha\langle v,\hat{x}\rangle A\hat{x} + Av)$.

In addition, if $\tan\rho_u(0) \le \frac{\alpha}{2}$, *then:*

(2)

$$\tan\rho_u(t) \le \begin{cases} \tan\rho_u(0)e^{-J(0,t)} + \frac{\alpha}{2}e^{-J(t,T_1)} & t\in[0,T_1],\\ (\tan\rho_u(T_1) + \alpha J(T_1,t))e^{-J(T_1,t)} & t\in[T_1,T] \end{cases} \quad \textit{and}$$

(3)

$$\int_{t_1}^{t_2} \|x\|^\alpha \tan\rho_u dt \le C \textit{ for all } 0<t_1<t_2<T.$$

Remark 5.2. *Note that by Statement 2 in Proposition 5.2 for any* $\epsilon > 0$ *we have that,*

$$\tan\rho_u(t) \le \begin{cases} \tan\rho_u(0)e^{-J(0,t)} + \frac{\alpha}{2}e^{-J(t,T_1)} & t\in[0,T_1],\\ C(\epsilon)e^{-(1-\epsilon)J(T_1,t)} & t\in[T_1,T] \end{cases} \tag{5.26}$$

In addition repeating the same argument as in [3] *with the reversed time and* $\tan\rho_s(T) \le \frac{\alpha}{2}$ *we obtain that if* $v(T) \in E^s_g(x(T))$*, then*

$$\int_{t_1}^{t_2} \|x\|^\alpha \tan\rho_s dt \le C. \tag{5.27}$$

We are now ready to estimate the contraction of stable vectors in Y. Let $v(0) \in E^s_g(x(0))$.

By Statement 1 in Proposition 5.2 we have that,

$$\begin{aligned} \log\left(\frac{\|v(t)\|}{\|v(0)\|}\right) &= \int_0^t \frac{1}{\|v(\tau)\|^2} \langle v(\tau), \dot v(\tau)\rangle \, d\tau \\ &= \int_0^t \langle \hat v(\tau), \|x(\tau)\|^\alpha(\alpha \langle \hat v(\tau), \hat x(\tau)\rangle A\hat x(\tau) + A\hat v(\tau))\rangle \, d\tau, \end{aligned} \tag{5.28}$$

where $\hat v = \frac{v}{\|v\|}$.

To make calculations simpler we drop the hat and just assume that for a fixed τ the norm $\|v(\tau)\| = 1$. We then have,

$$\begin{aligned} \langle v, \hat x\rangle \langle A\hat x, v\rangle &= (\langle v_s, \hat x_s\rangle + \langle v_u, \hat x_u\rangle)(-\beta \langle \hat x_s, v_s\rangle + \gamma \langle \hat x_u, v_u\rangle) \\ &= -\beta \langle \hat x_s, v_s\rangle^2 + \gamma \langle \hat x_u, v_u\rangle^2 + (\gamma - \beta) \langle \hat x_s, v_s\rangle \langle \hat x_u, v_u\rangle \le C \tan\rho_s, \end{aligned}$$

where we estimated

$$\langle \hat x_u, v_u\rangle \le \|v_u\| \le \frac{\|v_u\|}{\|v_s\|} = \tan\rho_s.$$

At the same time we have that,

$$\langle Av, v\rangle = -\beta\|v_s\|^2 + \gamma\|v_u\|^2 < -\beta + \lambda\|v_u\|^2 \le -\beta + C\tan\rho_s.$$

Together we have,

$$\langle v, \dot v\rangle \le -\beta\|x\|^\alpha + C\|x\|^\alpha \tan\rho_s.$$

By (5.25), (5.27) and (5.28) we have that

$$\log\left(\frac{\|v(t)\|}{\|v(0)\|}\right) \le C - \frac{1}{\alpha}\log(t+1). \tag{5.29}$$

In particular, for $y \in W^s_x$ we have that,

$$d(x(t), y(t)) \le C(t+1)^{-\frac{1}{\alpha}} d(x(0), y(0)). \tag{5.30}$$

Let now $y \in W^s_x$, $v \in E^u_g(x)$ and $w \in E^u_g(y)$. Define a number $s \in (-1, 1)$ such that $\|y(s)\| = \|x(0)\|$. Without loss of generality we assume that $\tan\theta(y,s) \ge \tan\theta(x,0)$. Let also $T_1 > 0$ be such that $\tan\theta(x, T_1) = 0$. We have the following.

Lemma 5.6. *There exists* $C > 0$ *such that:*

(1) $\frac{1}{\tan\theta(y,t)} \le C\frac{1}{\tan\theta(x,t)}$ *for* $t \le T_1$,
(2) $\tan\theta(y,t) \le C\tan\theta(x,t)$ *for* $t > T_1$,
(3) $\|y(t)\| \le C\|x(t)\|$ *for* $t \in (0, T)$.

Proof. First note that if $\|x_u(0)\| > c$ for some fixed positive number c, then the quantities: $\tan\theta(x,t)$, $\tan\theta(x,t)^{-1}$, $\tan\theta(y,t)$, $\tan\theta(y,t)^{-1}$, $\|x(t)\|$, $\|x(t)\|^{-1}$, $\|y(t)\|$, $\|y(t)\|^{-1}$, are all bounded uniformly for all $t \in (0, T)$ and the lemma follows.

Now assume that $\|x_u\| < c$ for some small $c > 0$ so that for some $\delta > 0 : \{x(t)| - \delta < t < \delta\} \subset K^s_\rho$ and $\{x(t)|T - \delta < t < T + \delta\} \subset K^u_\rho$.

Note that by conformality of the stable direction there exists a point $\tilde{y} = \tilde{y}(0)$ that lies on the plane containing the trajectory $\{x(t)|0 < t < T\}$ and such that $\|y(t)\| = \|\tilde{y}(t)\|$, and $\tan\theta(y,t) = \tan\theta(\tilde{y},t)$ for $t \in (0,T)$. In particular, since $\tan\theta(y,s) \geq \tan\theta(x,0)$, we have that $\|\tilde{y}_u(s)\| \leq \|x_u(0)\|$ and the interval I with $\tilde{y}(s)$ and $x(0)$ as endpoints is contained in the unstable cone. The same must be true for the image of I under the flow. Consequently, $\tan\theta(y,t) \geq C^{-1}\tan\theta(\tilde{y},t+s) \geq C^{-1}\tan\theta(x,t)$ for all $t \in (0,T_1)$, where $C = C(s) > 0$.

To show Statement (2) note that there exists $\tilde{s} > 0$ such that the point $\tilde{y}(s+\tilde{s}) \in W^s_{x(0)}$. To see this consider a small ball B around $x(T)$. Note that every piece of trajectory of the flow contained in B is contained in K^u_ρ and hence intersects $W^s_{x(T)}$ by transversality. Then we obtain that $\tan\theta(y,t) \leq C\tan\theta(\tilde{y},t+s+\tilde{s}) \leq C\tan\theta(x,t)$ for all $t \in (T_1,T)$, where $C = C(s,\tilde{s}) > 0$.

For Statement (3) we claim that $\|\tilde{y}(t+s)\| \leq \|x(t)\|$ for all $t \in (0,T)$. First observe that since $\|\tilde{y}(s)\| = \|x(0)\|$ and $\tan\theta(\tilde{y},t+s) \geq \tan\theta(x,t)$ for all $t \in (0,T_1)$, then by Statement (2) in Proposition 5.1 we obtain that $\|\tilde{y}(t+s)\| \leq \|x(t)\|$ for all $t \in (0,T_1)$. On the other hand, if $t > T_1$, we use the fact that the interval connecting $\tilde{y}(t+s)$ and $x(t)$ is contained in the unstable cone. Therefore, if $\tan\theta(\tilde{y},t+s) < \tan\theta(x,t)$, then necessarily $\|\tilde{y}(t+s)\| < \|x(t)\|$. Consequently, $\|\tilde{y}(t+s)\| \leq \|x(t)\|$ for $t \in (0,T)$.

We conclude that $\|y(t)\| \leq C\|x(t)\|$. □

From now on let ρ_u be the bigger of the two angles $\angle(v,E^u(0))$, $\angle(w,E^u(0))$ and let θ be the positive angle between the vector $\vec{0x}$ and $E^u(0)$.

Our goal is to find an upper bound for following quantity:

$$\log\left(\frac{\|v(T)\|}{\|w(T)\|}\right) = \log\left(\frac{\|v(0)\|}{\|w(0)\|}\right) + \int_0^T \left(\frac{1}{\|v\|^2}\langle v,\dot{v}\rangle - \frac{1}{\|w\|^2}\langle w,\dot{w}\rangle\right)dt \tag{5.31}$$

As before we may assume that for a fixed t the norms $\|v(t)\|, \|w(t)\| = 1$ so that the integrand on the right hand side of (5.31) simplifies to $(\langle v,\dot{v}\rangle - \langle w,\dot{w}\rangle)$.

Lemma 5.7. *We have the following two estimates of* $|\langle v,\dot{v}\rangle - \langle w,\dot{w}\rangle|$*:*

(5.32)
For all $t \leq T_1$,

$$|\langle v,\dot{v}\rangle - \langle w,\dot{w}\rangle| \leq C\|x\|^\alpha(\tan\rho_u + \frac{1}{\tan\theta})\|v-w\| + C\|y\|^{\alpha-1}(\tan\rho_u + \frac{1}{\tan\theta})\|x-y\| + C\,|\,\|x\|^\alpha - \|y\|^\alpha\,|;$$

(5.33)
and for all $T_1 \leq t \leq T$,

$$|\langle v,\dot{v}\rangle - \langle w,\dot{w}\rangle| \leq C\|x\|^\alpha(\tan\rho_u + \tan\theta)\|v-w\| + C\|y\|^{\alpha-1}(\tan\rho_u + \tan\theta)\|x-y\| + C\,|\,\|x\|^\alpha - \|y\|^\alpha\,|.$$

Proof of Lemma 5.7. We rewrite,

$$|\langle v,\dot{v}\rangle - \langle w,\dot{w}\rangle| \leq |\langle v-w,\dot{v}\rangle| + |\langle w,\dot{v}-\dot{w}\rangle| \tag{5.34}$$

By Statement 1 in Proposition 5.2, we have that

$$\begin{aligned} |\langle v-w,\dot{v}\rangle| &= |\langle v-w, \|x\|^\alpha(\alpha\langle v,\hat{x}\rangle A\hat{x} + Av)\rangle| \\ &\leq \|x\|^\alpha\left(|\langle v-w,A\hat{x}\rangle|\,|\langle v,\hat{x}\rangle|\alpha + |\langle v-w,Av\rangle|\right). \end{aligned} \tag{5.35}$$

We rewrite

$$| \langle v, \hat{x} \rangle | \leq | \langle v_s, \hat{x}_s \rangle | + | \langle v_u, \hat{x}_u \rangle |. \tag{5.36}$$

Note that $| \langle v_s, \hat{x}_s \rangle | \leq \|v_s\| \leq \frac{\|v_s\|}{\|v_u\|} \leq \tan\rho_u$, and $| \langle v_u, \hat{x}_u \rangle | \leq \|\hat{x}_u\| \leq \frac{\|\hat{x}_u\|}{\|\hat{x}_s\|} = \frac{1}{\tan\theta}$. Then we continue in (5.36),

$$| \langle v, \hat{x} \rangle | \leq \tan\rho_u + \frac{1}{\tan\theta}. \tag{5.37}$$

To estimate the remaining terms in (5.35) observe that,

$$\left| \left\langle \frac{(v-w)_s}{\|v-w\|}, \hat{x}_s \right\rangle \right| \leq \|\hat{x}_s\| \leq \frac{\|\hat{x}_s\|}{\|\hat{x}_u\|} = \tan\theta.$$

In addition, since the unit vectors v, w are within the angle ρ_u from $E^u(0)$, the vector $(v-w)$ is within the angle ρ_u from $E^s(0)$. This allows us to write,

$$\left| \left\langle \frac{(v-w)_u}{\|v-w\|}, \hat{x}_u \right\rangle \right| \leq \frac{\|(v-w)_u\|}{\|v-w\|} \leq \frac{\|(v-w)_u\|}{\|(v-w)_s\|} \leq \tan\rho_u.$$

By the definition of A, we then have that,

$$\begin{aligned} | \langle v-w, A\hat{x} \rangle | &\leq \beta \left| \left\langle \frac{(v-w)_s}{\|v-w\|}, \hat{x}_s \right\rangle \right| \; \|v-w\| + \gamma \left| \left\langle \frac{(v-w)_u}{\|v-w\|}, \hat{x}_u \right\rangle \right| \; \|v-w\| \\ &\leq C(\tan\theta + \tan\rho_u)\|v-w\|. \end{aligned} \tag{5.38}$$

Similarly, estimating $\|v_s\|$ by $\tan\rho_u$, we obtain that

$$\begin{aligned} | \langle v-w, Av \rangle | &\leq \beta \left| \left\langle \frac{(v-w)_s}{\|v-w\|}, v_s \right\rangle \right| \; \|v-w\| + \gamma \left| \left\langle \frac{(v-w)_u}{\|v-w\|}, v_u \right\rangle \right| \; \|v-w\| \\ &\leq C\tan\rho_u\|v-w\|. \end{aligned} \tag{5.39}$$

Substituting (5.37) and (5.39) in (5.35), and estimating $| \langle v-w, A\hat{x} \rangle | \leq C\|v-w\|$, we obtain that

$$| \langle v-w, \dot{v} \rangle | \leq C\|x\|^\alpha(\tan\rho_u + \frac{1}{\tan\theta})\|v-w\|. \tag{5.40}$$

On the other hand, substituting (5.38) and (5.39) in (5.35), and estimating $| \langle v, \hat{x} \rangle | \leq 1$, gives that

$$| \langle v-w, \dot{v} \rangle | \leq C\|x\|^\alpha(\tan\rho_u + \tan\theta)\|v-w\|. \tag{5.41}$$

We will use (5.40) to obtain (5.32) and (5.41) to obtain (5.33). To estimate the second term in (5.34) we calculate by Statement 1 in Proposition 5.2,

$$\begin{aligned} \dot{v} - \dot{w} &= \|x\|^\alpha(\alpha \langle v, \hat{x} \rangle A\hat{x} + Av) - \|y\|^\alpha(\alpha \langle w, \hat{y} \rangle A\hat{y} + Aw) \\ &= \|x\|^\alpha(\alpha \langle v, \hat{x} \rangle A\hat{x} - \alpha \langle w, \hat{y} \rangle A\hat{y} + A(v-w)) + (\|x\|^\alpha - \|y\|^\alpha)(\alpha \langle w, \hat{y} \rangle A\hat{y} + Aw). \end{aligned} \tag{5.42}$$

We can rewrite,

$$\begin{aligned} \langle v, \hat{x} \rangle A\hat{x} - \langle w, \hat{y} \rangle A\hat{y} &= \langle v-w, \hat{x} \rangle A\hat{x} + \langle w, \hat{x} \rangle A\hat{x} - \langle w, \hat{y} \rangle A\hat{y} \\ &= \langle v-w, \hat{x} \rangle A\hat{x} + \langle w, \hat{x} - \hat{y} \rangle A\hat{x} + \langle w, \hat{y} \rangle A(\hat{x} - \hat{y}). \end{aligned} \tag{5.43}$$

Substituting (5.43) in (5.42) we obtain that,

$$\begin{aligned} \dot{v} - \dot{w} &= \|x\|^\alpha\alpha \left(\langle v-w, \hat{x} \rangle A\hat{x} + \langle w, \hat{x} - \hat{y} \rangle A\hat{x} + \langle w, \hat{y} \rangle A(\hat{x} - \hat{y})\right) \\ &\quad + \|x\|^\alpha A(v-w) + (\|x\|^\alpha - \|y\|^\alpha)(\alpha \langle w, \hat{y} \rangle A\hat{y} + Aw). \end{aligned} \tag{5.44}$$

Consequently,

$$
\begin{aligned}
|\langle \dot v - \dot w, w\rangle| \le \|x\|^\alpha \alpha\,(|\langle v-w,\hat x\rangle|\,|\langle A\hat x, w\rangle| + |\langle w, \hat x - \hat y\rangle|\,|\langle A\hat x, w\rangle| \\
+ |\langle w,\hat y\rangle|\,|\langle A(\hat x-\hat y), w\rangle|\,) \\
+ \|x\|^\alpha |\langle A(v-w), w\rangle| + |\,\|x\|^\alpha - \|y\|^\alpha\,|\,(\alpha|\langle w,\hat y\rangle|\,|\langle A\hat y, w\rangle| + |\langle Aw, w\rangle|).
\end{aligned}
\tag{5.45}
$$

We estimate term by term,

$$
|\langle A\hat x, w\rangle| \le \beta|\langle \hat x_s, w_s\rangle| + \gamma|\langle \hat x_u, w_u\rangle| \le C(\tan\rho_u + \frac{1}{\tan\theta}),
\tag{5.46}
$$

where we estimated

$$
\begin{aligned}
|\langle \hat x_s, w_s\rangle| \le \|w_s\| \le \frac{\|w_s\|}{\|w_u\|} \le \tan\rho_u, \text{and} \\
|\langle \hat x_u, w_u\rangle| \le \|\hat x_u\| \le \frac{\|\hat x_u\|}{\|\hat x_s\|} = \frac{1}{\tan\theta}.
\end{aligned}
$$

Similarly, by Lemma 5.6, we have for $t \le T_1$,

$$
|\langle w, \hat y\rangle| \le |\langle w_s, \hat y_s\rangle| + |\langle w_u, \hat y_u\rangle| \le \tan\rho_u + C\frac{1}{\tan\theta}.
\tag{5.47}
$$

As in (5.38) we can write,

$$
\begin{aligned}
|\langle v-w, \hat x\rangle| &\le \left|\left\langle \frac{(v-w)_s}{\|v-w\|}, \hat x_s\right\rangle\right|\ \|v-w\| + \left|\left\langle \frac{(v-w)_u}{\|v-w\|}, \hat x_u\right\rangle\right|\ \|v-w\| \\
&\le (\tan\theta + \tan\rho_u)\|v-w\|.
\end{aligned}
\tag{5.48}
$$

In a similar manner, estimating $\|w_s\|$ by $\tan\rho_u$, we obtain that

$$
\begin{aligned}
|\langle A(v-w), w\rangle| &\le \beta\left|\left\langle \frac{(v-w)_s}{\|v-w\|}, w_s\right\rangle\right|\ \|v-w\| + \gamma\left|\left\langle \frac{(v-w)_u}{\|v-w\|}, w_u\right\rangle\right|\ \|v-w\| \\
&\le C\tan\rho_u\|v-w\|.
\end{aligned}
\tag{5.49}
$$

To estimate the remaining terms note that,

$$
\left|\left\langle w_s, \frac{(\hat x-\hat y)_s}{\|\hat x-\hat y\|}\right\rangle\right| \le \|w_s\| \le \frac{\|w_s\|}{\|w_u\|} \le \tan\rho_u.
$$

Let now θ_{max} be such that the vectors $\vec{0x}, \vec{0y}$ are within the angle θ_{max} from $E^u(0)$. then the vector $(\vec{0\hat x} - \vec{0\hat y})$ is within the angle θ_{max} from $E^s(0)$ and Lemma 5.6 gives for $t \ge T_1$,

$$
\left|\left\langle w_u, \frac{(\hat x-\hat y)_u}{\|\hat x-\hat y\|}\right\rangle\right| \le \frac{\|(\hat x-\hat y)_u\|}{\|\hat x-\hat y\|} \le \frac{\|(\hat x-\hat y)_u\|}{\|(\hat x-\hat y)_s\|} \le \tan\theta_{\max} \le C\tan\theta.
$$

Finally, if $t \ge T_1$, we have that,

$$
\begin{aligned}
|\langle w, \hat x-\hat y\rangle| &\le \left|\left\langle w_s, \frac{(\hat x-\hat y)_s}{\|\hat x-\hat y\|}\right\rangle\right|\ \|\hat x-\hat y\| + \left|\left\langle w_u, \frac{(\hat x-\hat y)_u}{\|\hat x-\hat y\|}\right\rangle\right|\ \|\hat x-\hat y\| \\
&\le (\tan\rho_u + C\tan\theta)\|\hat x-\hat y\|, \text{ and}
\end{aligned}
\tag{5.50}
$$

$$
\begin{aligned}
|\langle A(\hat x-\hat y), w\rangle| &\le \beta\left|\left\langle \frac{(\hat x-\hat y)_s}{\|\hat x-\hat y\|}, w_s\right\rangle\right|\ \|\hat x-\hat y\| + \gamma\left|\left\langle \frac{(\hat x-\hat y)_u}{\|\hat x-\hat y\|}, w_u\right\rangle\right|\ \|\hat x-\hat y\| \\
&\le C(\tan\rho_u + \tan\theta)\|\hat x-\hat y\|.
\end{aligned}
\tag{5.51}
$$

Substituting (5.46), (5.47) and (5.49) in (5.45) gives, for $t \leq T_1$,

(5.52)
$$| \langle \dot{v} - \dot{w}, w \rangle | \leq C\|x\|^\alpha (\tan\rho_u + \frac{1}{\tan\theta})\|v-w\| + C\|x\|^\alpha (\tan\rho_u + \frac{1}{\tan\theta})\|\hat{x}-\hat{y}\| + C \mid \|x\|^\alpha - \|y\|^\alpha \mid .$$

on the other hand, substituting (5.48), (5.49), (5.50) and (5.51) in (5.45) we obtain, for $t \geq T_1$,

$$| \langle \dot{v} - \dot{w}, w \rangle | \leq C\|x\|^\alpha (\tan\rho_u + \tan\theta)\|v-w\| + C\|x\|^\alpha (\tan\rho_u + \tan\theta)\|\hat{x}-\hat{y}\| + C \mid \|x\|^\alpha - \|y\|^\alpha \mid . \tag{5.53}$$

The lemma follows by substituting (5.40), (5.41), (5.52) and (5.53) in (5.34) and observing that

$$\|\hat{x} - \hat{y}\| \leq C \frac{\|x-y\|}{\|y\|}.$$

□

We now use Lemma 5.7 to find an upper estimate for the integral in (5.31). Using estimates from Lemma 5.6 in the proof of Lemma 7.8 in [3] one can easily see that $\tan\rho_u$ can be estimated as in (5.26). Consequently, using Statement 1 in Proposition 5.1 and (5.26) we can write for $t < T_1$,

$$\tan\rho_u(t) + \frac{1}{\tan\theta(t)} = \tan\rho_u(t) + \frac{\tan\theta(T_1)}{\tan\theta(t)} \leq e^{-J(0,t)} + Ce^{-J(t,T_1)}. \tag{5.54}$$

On the other hand, for $t \geq T_1$ we have that,

$$\tan\rho_u(t) + \tan\theta(t) = \tan\rho_u(t) + \frac{\tan\theta(t)}{\tan\theta(T_1)} \leq Ce^{-(1-\epsilon)J(T_1,t)}. \tag{5.55}$$

Define

$$c(t) =: \begin{cases} e^{-J(0,t)} + Ce^{-J(t,T_1)} & t < T_1 \\ Ce^{-(1-\epsilon)J(T_1,t)} & t > T_1. \end{cases} \tag{5.56}$$

By Lemma 5.7, the integral in (5.31) can be estimated as (recall that we assume $\|v\| = \|w\| = 1$),

$$\mathcal{J}_1 + \mathcal{J}_2 + \mathcal{J}_3, \tag{5.57}$$

with

$$\mathcal{J}_1 := C\int_0^T \|x\|^\alpha c(t)\|v-w\|dt, \tag{5.58}$$

$$\mathcal{J}_2 := C\int_0^T \|y\|^{\alpha-1} c(t)\|x-y\|dt, \tag{5.59}$$

$$\mathcal{J}_3 := C\int_0^T \mid \|x\|^\alpha - \|y\|^\alpha \mid dt. \tag{5.60}$$

We start by estimating $\mathcal{J}_1$. Define $\eta(t) := \|v(t) - w(t)\|$. The following proposition describes how η changes.

Proposition 5.3. *([3], Lemma 7.11)*

$$\eta' \leq (C\tan\rho_u\|x\|^\alpha - \gamma\|x\|^\alpha)\eta + C\|y\|^{\alpha-1}\|x-y\|.$$

Using the above proposition we obtain the following.

Lemma 5.8. $\mathcal{J}_1 \leq C\eta(0) + Cd(x(0), y(0))$.

Proof. Denoting $I(t_1,t_2) := \int_{t_1}^{t_2}(\gamma\|x\|^\alpha - C\tan\rho_u\|x\|^\alpha)dt$ we have that

$$\frac{d}{ds}\left(e^{I(0,s)}\eta(s)\right) = e^{I(0,s)}\left((\gamma\|x\|^\alpha - C\tan\rho_u\|x\|^\alpha)\eta + \eta'\right) \le e^{I(0,s)}\|y\|^{\alpha-1}\|x-y\|.$$

Integrating both sides from 0 to t gives

$$e^{I(0,t)}\eta(t) \le \eta(0) + \int_0^t e^{I(0,s)}\|y\|^{\alpha-1}\|x-y\|ds, \text{ so that}$$

$$\eta(t) \le \eta(0)e^{-I(0,t)} + \int_0^t e^{-I(s,t)}\|y\|^{\alpha-1}\|x-y\|ds. \tag{5.61}$$

By (5.23), (5.25), Statement 3 in Proposition 5.2 and (5.30), the integral on the right hand side can be estimated by

$$\begin{aligned}&C\int_0^t \left(\frac{t+1}{s+1}\right)^{\frac{-\gamma}{\beta\alpha}} (s+1)^{-1+\frac{1}{\alpha}}(s+1)^{\frac{-1}{\alpha}}d(x(0),y(0))ds\\ \le& C(t+1)^{\frac{-\gamma}{\beta\alpha}}(s+1)^{\frac{\gamma}{\beta\alpha}}|_0^t d(x(0),y(0))\\ \le& Cd(x(0),y(0)).\end{aligned} \tag{5.62}$$

Using (5.61) and (5.62) in (5.58) gives that,

$$\mathcal{J}_1 \le C(\eta(0) + d(x(0),y(0)))\int_0^T \|x\|^\alpha c(t)dt. \tag{5.63}$$

By (5.24) and (5.25), we have that,

$$\begin{aligned}\int_0^{T_1}\|x\|^\alpha c(t)dt &\le C\int_0^{T_1}(t+1)^{-1}(t+1)^{\frac{-\lambda}{\alpha\beta}}dt + C\int_0^{T_1}(t+1)^{-1}\left(\frac{T_1+1}{t+1}\right)^{\frac{-\lambda}{\alpha\beta}}dt\\ &\le C(t+1)^{\frac{-\lambda}{\alpha\beta}}|_{T_1}^0 + C(T_1+1)^{\frac{-\lambda}{\alpha\beta}}(t+1)^{\frac{\lambda}{\alpha\beta}}|_0^{T_1} \le C.\end{aligned} \tag{5.64}$$

To estimate $\int_{T_1}^T \|x\|^\alpha c(t)dt$ observe that

$$\frac{d}{dt}\left(e^{-(1-\epsilon)J(T_1,t)}\right) = -(1-\epsilon)\lambda e^{-(1-\epsilon)J(T_1,t)}\|x\|^\alpha. \text{ Consequently,}$$

$$\int_{T_1}^T \|x\|^\alpha c(t)dt = -\frac{1}{(1-\epsilon)\lambda}(e^{-(1-\epsilon)J(T_1,T)} - 1) \le C. \tag{5.65}$$

The lemma follows if we substitute (5.64) and (5.65) in (5.63). ☐

We now estimate $\mathcal{J}_2$.

Lemma 5.9. $\mathcal{J}_2 \le Cd(x(0),y(0))$.

Proof. By (5.23) and (5.30), we have that

$$\mathcal{J}_2 \le C\int_0^T c(t)(t+1)^{-1}d(x(0),y(0))dt.$$

Using (5.25), we obtain that

$$\begin{aligned}\int_0^{T_1} c(t)(t+1)^{-1} d(x(0),y(0))dt &\le \int_0^{T_1} (t+1)^{-1-\frac{\lambda}{\alpha\beta}} d(x(0),y(0))dt \\ &+ \int_0^{T_1} (T_1+1)^{-\frac{\lambda}{\alpha\beta}} (t+1)^{-1+\frac{\lambda}{\alpha\beta}} d(x(0),y(0))dt \\ &\le Cd(x(0),y(0)). \text{ Similarly,} \\ \int_{T_1}^{T} c(t)(t+1)^{-1} d(x(0),y(0))dt &\le \int_{T_1}^{T} (T_1+1)^{\frac{\tilde{\lambda}}{\alpha\beta}} (t+1)^{-1-\frac{\tilde{\lambda}}{\alpha\beta}} d(x(0),y(0))dt \\ &\le Cd(x(0),y(0)).\end{aligned}$$

The lemma follows. □

It remains to estimate $\mathcal{J}_3$. We have the following.

Lemma 5.10. $\mathcal{J}_3 \le Cd(x(0),y(0))$.

Proof. Rewrite,

$$\int_0^T |\,\|x\|^\alpha - \|y\|^\alpha| dt = \int_{\{t|\ \|x(t)\|\ge\|y(t)\|\}} \|x\|^\alpha - \|y\|^\alpha dt + \int_{\{t|\ \|x(t)\|\le\|y(t)\|\}} \|y\|^\alpha - \|x\|^\alpha dt.$$

We use Statement 1 in Proposition 5.1 to write,

$$\int_{t_1}^{t_2} \|x\|^\alpha dt = \lambda^{-1} \log\left(\frac{\tan\theta(t_1)}{\tan\theta(t_2)}\right).$$

First assume that $\|x_u(0)\| < c$ for $c > 0$ as in the proof of Lemma 5.6. Let also $\tilde{y}$, s and $\tilde{s}$ be as in the proof of Lemma 5.6. Say $\|x(t)\| \ge \|y(t)\|$ for $t_1 < t < t_2$. Then using the fact that $\|\tilde{y}(t+s)\| \le \|x(t)\|$ we can estimate,

$$\begin{aligned}&\int_{t_1}^{t_2} \|x(t)\|^\alpha - \|y(t)\|^\alpha dt = \int_{t_1}^{t_2} \|x(t)\|^\alpha - \|\tilde{y}(t+s)\|^\alpha dt + \int_{t_1}^{t_2} \|\tilde{y}(t+s)\|^\alpha - \|\tilde{y}(t)\|^\alpha dt \\ &\le \int_0^T \|x(t)\|^\alpha - \|\tilde{y}(t+s)\|^\alpha dt + \int_{t_1}^{t_2} \|\tilde{y}(t+s)\|^\alpha - \|\tilde{y}(t)\|^\alpha dt \\ &= \lambda^{-1}\log\left(\frac{\tan\theta(x,0)}{\tan\theta(\tilde{y},s)}\frac{\tan\theta(\tilde{y},T+s)}{\tan\theta(x,T)}\right) + \int_{t_1+s}^{t_2+s} \|\tilde{y}(t)\|^\alpha dt - \int_{t_1}^{t_2} \|\tilde{y}(t)\|^\alpha dt \\ &= \lambda^{-1}\log\left(\frac{\tan\theta(x,0)}{\tan\theta(\tilde{y},s)}\frac{\tan\theta(\tilde{y},T+s+\tilde{s})}{\tan\theta(x,T)}\frac{\tan\theta(\tilde{y},T+s)}{\tan\theta(\tilde{y},T+s+\tilde{s})}\right) + \int_{t_2}^{t_2+s} \|\tilde{y}(t)\|^\alpha dt - \int_{t_1}^{t_1+s} \|\tilde{y}(t)\|^\alpha dt \\ &\le |\tilde{s}| + 2|s| \le Cd(x,y). \text{ We used the fact that } \tan\theta(x,0) \le \tan\theta(\tilde{y},s),\ \tan\theta(\tilde{y},T+s+\tilde{s}) \le \tan\theta(x,T).\end{aligned}$$

Let now $\|x(t)\| \le \|y(t)\|$ for $t_1 < t < t_2$. Using the fact that $\|\tilde{y}(t+s)\| \le \|x(t)\|$ we can estimate

$$\begin{aligned}\int_{t_1}^{t_2} \|y(t)\|^\alpha - \|x(t)\|^\alpha dt &= \int_{t_1}^{t_2} \|\tilde{y}(t+s)\|^\alpha - \|x(t)\|^\alpha dt + \int_{t_1}^{t_2} \|\tilde{y}(t)\|^\alpha - \|\tilde{y}(t+s)\|^\alpha dt \\ &\le \int_{t_1}^{t_2} \|\tilde{y}(t)\|^\alpha - \|\tilde{y}(t+s)\|^\alpha dt \le 2|s| \le Cd(x,y).\end{aligned}$$

We now consider a case when $\|x_u(0)\| > c$. This implies that there exists $\tilde{c} = \tilde{c}(c, d(x,y))$ such that for all $t \in (0,T)$ the quantities: $\|x_u(t)\|$, $\|x_s(t)\|$, $\|y_u(t)\|$, $\|y_s(t)\|$ are greater than $\tilde{c}$. Then for all $t \in (0,T)$ we have that,

$$\begin{aligned}\log \frac{\tan\theta(x)}{\tan\theta(y)} &= \log\left(\frac{\|x_s\|}{\|x_u\|}\frac{\|y_u\|}{\|y_s\|}\right)\\ &\leq \log \frac{\|y_s\| + \|x_s - y_s\|}{\|y_s\|} + \log \frac{\|x_u\| + \|y_u - x_u\|}{\|x_u\|}\\ &\leq \log\left(1 + \frac{\|x_s - y_s\|}{\|y_s\|}\right) + \log\left(1 + \frac{\|y_u - x_u\|}{\|x_u\|}\right)\\ &\leq C\|x_s - y_s\| + C\|y_u - x_u\| \leq Cd(x,y).\end{aligned}$$

Similarly, $\log \frac{\tan\theta(y)}{\tan\theta(x)} \leq Cd(x,y)$. The lemma follows. □

We are now ready to give the upper bound for the quantity (5.31). By (5.57) and the Lemmas 5.8, 5.9 and 5.10, we may write the following.

Lemma 5.11. *For $y \in W^s_x$ and two tangent vectors $v \in E^u_g(x), w \in E^u_g(y)$ we have that,*

$$\log \frac{\|v(T)\|}{\|w(T)\|} \leq \log \frac{\|v(0)\|}{\|w(0)\|} + C\eta(0) + Cd(x(0), y(0)), \tag{5.66}$$

where $\eta(0) = \|\frac{v(0)}{\|v(0)\|} - \frac{w(0)}{\|w(0)\|}\|$ and x enters $B(0, r_1)$ at $t = 0$ and exits at $t = T$.

We now want to use Lemma 5.11 to prove $(A3)(a)$. Fix $x \in P$, $y \in W^s_x$ and consider two unit tangent vectors, $v \in E^u_g(x)$ and $w \in E^u_g(y)$. Let $0 \leq n \leq N := \tau(x)$, where $\tau(x) < \infty$ is the first return time to P. Denote:

$$x_n := g^n(x),\ y_n := g^n(y);\ v_n := (Dg^n(x))(v),\ w_n := (Dg^n(y))(w);$$
$$\hat{v}_n := \frac{v_n}{\|v_n\|},\ \hat{w}_n := \frac{w_n}{\|w_n\|};\ a_n := \log \frac{\|(Dg^n(x))(v)\|}{\|(Dg^n(y))(w)\|}.$$

We have that,

$$\begin{aligned}a_{n+1} &= \log \frac{\|(Dg^{n+1}(x))(v)\|}{\|(Dg^{n+1}(y))(w)\|} = \log \frac{\|(Dg^n(x))(v)\|}{\|(Dg^n(y))(w)\|} + \log \frac{\frac{\|(Dg^{n+1}(x))(v)\|}{\|(Dg^n(x))(v)\|}}{\frac{\|(Dg^{n+1}(y))(w)\|}{\|(Dg^n(y))(w)\|}}\\ &= a_n + \log \frac{\|(Dg(x_n))(\hat{v}_n)\|}{\|(Dg(y_n))(\hat{w}_n)\|} = a_n + \log \frac{\|(Dg(x_n))(\hat{v}_n)\|}{\|(Dg(x_n))(\hat{w}_n)\|} + \log \frac{\|(Dg(x_n))(\hat{w}_n)\|}{\|(Dg(y_n))(\hat{w}_n)\|}.\end{aligned} \tag{5.67}$$

Using the fact that g is $C^{1+\alpha}$, the second term in (5.67) can be estimated as follows,

$$\begin{aligned}\log \frac{\|(Dg(x_n))(\hat{v}_n)\|}{\|(Dg(x_n))(\hat{w}_n)\|} &\leq \log \frac{\|(Dg(x_n))(\hat{w}_n)\| + \|(Dg(x_n))(\hat{v}_n - \hat{w}_n)\|}{\|(Dg(x_n))(\hat{w}_n)\|}\\ &= \log\left(1 + \frac{\|(Dg(x_n))(\hat{v}_n - \hat{w}_n)\|}{\|(Dg(x_n))(\hat{w}_n)\|}\right) \leq C\|(\hat{v}_n - \hat{w}_n)\|.\end{aligned} \tag{5.68}$$

In addition, the last term in (5.67) can be estimated by,

$$\begin{aligned}\log \frac{\|(Dg(x_n))(\hat{w}_n)\|}{\|(Dg(y_n))(\hat{w}_n)\|} &\leq \log \frac{\|(Dg(y_n))(\hat{w}_n)\| + \|(Dg(x_n) - Dg(y_n))(\hat{w}_n)\|}{\|(Dg(y_n))(\hat{w}_n)\|}\\ &\leq \log\left(1 + \frac{\|(Dg(x_n) - Dg(y_n))(\hat{w}_n)\|}{\|(Dg(y_n))(\hat{w}_n)\|}\right) \leq Cd(x_n, y_n)^\alpha.\end{aligned} \tag{5.69}$$

Substituting (5.68) and (5.69) in (5.67) gives,

$$a_{n+1} \leq a_n + C\|(\hat{v}_n - \hat{w}_n)\| + Cd(x_n, y_n)^\alpha. \tag{5.70}$$

Given $x \in \Lambda_g$, define finite collections of positive integers: $\{n_i \mid 0 \leq i \leq k\}$ and $\{Q_i \mid 0 \leq i \leq k\}$ such that $0 = n_0 < n_1 < \ldots < n_k < N$, $n_k + Q_k = N$, and $x_n \in B(0, r_1)$ if and only if $n_i + Q_i < n < n_{i+1}$ for some $0 \leq i < k$. By Remark 5.1, for every $i \in \{0, \ldots, k\}$ we have that $Q_i > Q$ for some large Q.

Denoting $\eta_n := \|(\hat{v}_n - \hat{w}_n)\|$ and $d_n := d(x_n, y_n)^\alpha$, Lemma 5.11 and (5.70) give that

$$a_N \leq \sum_{i=0}^{k} \left(C \sum_{n=n_i}^{n_i+Q_i} (\eta_n + d_n) \right) = \sum_{i=0}^{k} \left(C \sum_{l=0}^{Q_i} (\eta_{n_i+l} + d_{n_i+l}) \right). \tag{5.71}$$

We have the following.

Lemma 5.12. *If Q is sufficiently large then there exist $s > 1$ such that for all $i \in \{0, \ldots, k\}$ and for all $l \in \{0, \ldots, Q_i\}$ the following holds:*

- *(1)* $\eta_{n_i+l} \leq Cs^{-l}(\eta_{n_i} + d_{n_i})$,
- *(2)* $\eta_{n_i} \leq s^{-i}(\eta_0 + d_0)$,
- *(3)* $d_{n_i+l} \leq Cs^{-l}d_{n_i}$,
- *(4)* $d_{n_i} \leq s^{-i}d_0$.

We first show how (A3)(a) follows from the above lemma. Using *(1)*, *(3)* and (5.71) gives that

$$a_N \leq \sum_{i=0}^{k} \left(C \sum_{l=0}^{Q_i} (\eta_{n_i+l} + d_{n_i+l}) \right) \leq \sum_{i=0}^{k} C \sum_{l=0}^{Q_i} s^{-l} \left(\eta_{n_i} + d_{n_i}\right) \leq \sum_{i=0}^{k} C \left(\eta_{n_i} + d_{n_i}\right).$$

It follows from Statements *(2)* and *(4)* that

$$\leq \sum_{i=0}^{k} Cs^{-i} \left(\eta_0 + d_0\right). \leq C(\eta_0 + d_0). \tag{5.72}$$

Consider a sequence of positive integers, $\{N_K\}_{K\geq 1}$ defined in such a way that $g^{N_K}(x) = F^K(x)$. By (5.72), we have that

$$\log \frac{J^u F(F^K(x))}{J^u F(F^K(y))} \leq C(\eta_{N_{K-1}} + d_{N_{K-1}}).$$

Then using Statements *(2)* and *(4)*, we obtain that

$$\leq Cs^{-K}(\eta_0 + d_0) \leq Cs^{-K}.$$

This concludes the proof of $(A3)(a)$ up to the proof of Lemma 5.12. □

Proof of Lemma 5.12. Note that outside of $B(0, r_1)$ stable curves are uniformly contracted, which implies *(3)*. Then *(4)* follows by (5.30) if Q is large enough. We now show *(1)*. For a fixed $i \in \{0, \ldots, k\}$ and $l \in \{0, \ldots, Q_i\}$ denote $n := n_i + l$.

Observe that $C^{-1}\angle(v_n, w_n) \le \eta_n \le C\angle(v_n, w_n)$. In addition,

$$\begin{aligned}\angle(v_{n+1}, w_{n+1}) = \angle(Dg(x_n)(v_n), Dg(y_n)(w_n)) &\le \angle(Dg(x_n)(v_n), Dg(x_n)(w_n)) + Cd(x_n, y_n)^\alpha \\ &\le s^{-1}\angle(v_n, w_n) + Cd(x_n, y_n)^\alpha\end{aligned}$$

for some $s > 1$. Consequently,

$$\eta_{n_i+l} \le C\eta_{n_i} s^{-l} + Cd_{n_i}\nu^{-\alpha l}. \tag{5.73}$$

This proves Statement *(1)*. To show Statement *(2)* observe that by (5.61) and (5.62), noting that $d(x, y) \le d(x, y)^\alpha$ for $d(x, y) \le 1$, we have that

$$\eta_{n_{i+1}} \le C(\eta_{n_i+Q_i} + d_{n_i+Q_i}). \tag{5.74}$$

By Statement *(1)*,

$$\eta_{n_i+Q_i} \le Cs^{-Q_i}(\eta_{n_i} + d_{n_i}). \tag{5.75}$$

By Statement *(3)* we also have that

$$d_{n_i+Q_i} \le C(\nu^\alpha)^{-Q_i} d_{n_i}.$$

If now Q is large enough this implies that $\eta_{n_{i+1}} \le \tilde{s}(\eta_{n_i} + d_{n_i})$ for some $\tilde{s} < 1$ which in turn implies Statement *(2)*. □

5.3. **Proofs of Theorems A and B.**

Proof of Theorem A. Since the map g is expansive (see Theorem 3.1), it admits an equilibrium measure for any continuous potential. Consider the geometric potential $\varphi_{t,g} = -t\log|J^u g|$. By continuity property of $E^u_g(x)$ (see Theorem 3.3), we have that $\varphi_{t,g}$ is continuous for all t. □

Proof of Theorem B. By Theorem 4.2 and Proposition 4.1 for $t = 1$ there exist an equilibrium measure which is a unique SRB measure. By the Entropy formula, the pressure at $t = 1$ is 0. Clearly the Dirac measure at zero gives pressure zero for all $t \in \mathbb{R}$. Since the pressure is nonincreasing in t we have that Dirac measure at zero is an equilibrium measure for all $t \ge 1$. To see that it is the only equilibrium measure for $t > 1$ observe that by the Margulis-Ruelle inequality we have that $h_\mu(g) \le \int \log|J^u g(x)|d\mu$ for any invariant Borel probability measure μ. If now a measure μ is not equal to the Dirac measure at zero, it must give positive weight to the complement of $B(0, r_1)$ and hence $\int \varphi_1 d\mu < 0$. Then we have $h_\mu(g) + t\int \varphi_1 d\mu \le (t-1)\int \varphi_1 d\mu < 0$.

We need the following result for the remaining part of the proof.

Lemma 5.13. *Assume that f is a C^1-small perturbation of a certain local diffeomorphism $\bar{f}$, for which the SRB measure $\bar{\omega}_1$ and the measure of maximal entropy $\bar{\omega}_0$ coincide, and let $r_1 > 0$ be small enough. If the induced map F for the map g is constructed as in the proof of Theorem 4.2, then there exists $0 < h < -\int \varphi_1 d\mu_1$, where μ_1 is the SRB-measure for g, such that*

$$S_n = \#\{i \mid \tau_i = n\} \le Ce^{hn}. \tag{5.76}$$

Proof of Lemma 5.13. We need to estimate the number of sets P^s_i with a given i. This number is the same as the number of periodic orbits of g of minimal period τ_i that originate in P and do not return until time τ_i. Using the symbolic representation of g one can see that the latter equals the number of symbolic words of length τ_i for which the symbol P occurs only as the first and last symbol. The number of such words grows exponentially with exponent $0 < h < h_{top}(g) = h_{top}(f) = h_{top}(\bar{f})$.

We now show that $h < -\int \varphi_1 d\mu_1$. Since the SRB measure and the measure of maximal entropy for $\bar{f}$ coincide, by Proposition 20.3.10 in [6] we have that $\log|J^u\bar{f}| = u\circ\bar{f} - u + \log\lambda$ for some Hölder continuous $u : \Lambda_{\bar{f}} \to \mathbb{R}$ and $\log\lambda = h_{top}(\bar{f}) > 0$. Choose a Markov partition $\bar{\mathcal{P}}$ for $\bar{f}$ of small diameter. Let $h < h_{top}(\bar{f})$ be such that (5.76) holds for $\bar{f}$ and all $P \in \bar{\mathcal{P}}$. Let f be sufficiently close to $\bar{f}$ in C^1 topology, so that:

(1) there exists a homeomorphism $\bar{h} : \Lambda_f \to \Lambda_{\bar{f}}$ such that $\bar{h}\circ f = \bar{f}\circ\bar{h}$;
(2) There exists $0 < \epsilon < (h_{top} - h)/2$ such that for every $\tilde{x} \in \Lambda_f$ and $\bar{x} := \bar{h}(\tilde{x})$ we have, $|\ \log|J^u f(\tilde{x})| - \log|J^u\bar{f}(\bar{x})|\ | < \epsilon$.

Then for every $\tilde{x} \in \Lambda_f$ we have that

$$\log|J^u f(\tilde{x})| \geq u(\bar{f}(\bar{h}(\tilde{x}))) - u(\bar{h}(\tilde{x})) + h_{top}(\bar{f}) - \epsilon. \tag{5.77}$$

Denote $\kappa_0 := \max_{\bar{x}\in\Lambda_{\bar{f}}} u(\bar{x}) - \min_{\bar{x}\in\Lambda_{\bar{f}}} u(\bar{x})$.

Let now g be the slow down of f. Take $x \in \Lambda_g$ and denote $\tilde{x} := h(x) \in \Lambda$, $p := W^u_{\tilde{x},f} \cap W^s_{x,g} \in \Lambda$. We have that $|J^u f(\tilde{x})| = |Df(\tilde{x})(\tilde{v})|$, $|J^u f(p)| = |Df(p)(w)|$ and $|J^u g(x)| = |Dg(x)(v)|$, where $\tilde{v} \in E^u_f(\tilde{x})$, $w \in E^u_f(p)$, $v \in E^u_g(x)$ are unit vectors. Let $m \geq 1$ be such that $x, g(x), \dots g^m(x) \in U \setminus B(0, r_1)$. By distortion estimates presented in the previous section (see (5.70) and Statement (1) of Lemma 5.12), we have that

$$\left|\log\frac{\|Df^m(p)(w)\|}{\|Dg^m(x)(v)\|}\right| \leq C\|v - w\| + Cd(x,p)^\alpha \leq \kappa_1,$$

for some $\kappa_1 > 0$ which does not depend on m. At the same time, because $\log|J^u f|$ is a Hölder continuous function on a uniformly hyperbolic set Λ, we have that

$$\left|\log\frac{\|Df^m(\tilde{x})(\tilde{v})\|}{\|Df^m(p)(w)\|}\right| = |\ \sum_{k=0}^{m-1}\log|J^u f(f^k(\tilde{x}))| - \sum_{k=0}^{m-1}\log|J^u f(f^k(p))|\ | \leq \kappa_2,$$

for some $\kappa_2 > 0$ which does not depend on m. Together we have that

$$|\sum_{k=0}^{m-1}\log|J^u g(g^k(x))| - \sum_{k=0}^{m-1}\log|J^u f(f^k(\tilde{x}))|\ | \leq \kappa_1 + \kappa_2. \tag{5.78}$$

By (5.77) this implies that

$$\sum_{k=0}^{m-1}\log|J^u g(g^k(x))| \geq u(\bar{f}^m(\bar{h}(\tilde{x}))) - u(\bar{h}(\tilde{x})) + m(h_{top}(\bar{f}) - \epsilon) - \kappa_0 - \kappa_1 - \kappa_2.$$

On the other hand, for $x, g(x), \dots, g^m(x) \in B(0, r_1)$, by (3.4) in the Proposition 3.2 we have that,

$$\sum_{k=0}^{m-1}\log|J^u g(g^k(x))| \geq \log C_2 \text{ for some } C_2 > 0. \tag{5.79}$$

Denote $\kappa := \kappa_0 + \kappa_1 + \kappa_2$. We then have

$$\begin{aligned}\frac{1}{n}\sum_{m=0}^{n-1}\log|J^u g(g^m(x))| &= \frac{1}{n}\sum_{m=0}^{n-1}\log|J^u g(g^m(x))|\chi_{U\setminus B(0,r_1)}(g^m(x))\\ &+\frac{1}{n}\sum_{m=0}^{n-1}\log|J^u g(g^m(x))|\ \chi_{B(0,r_1)}(g^m(x))\\ &\geq \frac{1}{n}(h_{top}(\bar{f})-\epsilon)\sum_{m=0}^{n-1}\chi_{U\setminus B(0,r_1)}(g^m(x)) - \frac{k(n)\kappa}{n} + \frac{k(n)\log C_2}{n},\end{aligned} \tag{5.80}$$

where $\chi_A(x)$ denotes the characteristic function of a set A and $k(n)$ denotes the number of exits of the orbit of x from $B(0,r_1)$ up to time n. By Remark 5.1, we have that $k(n)Q \leq n+1$. Thus, $\frac{k(n)}{n} \leq \frac{1}{Q} + \frac{1}{Qn}$. By the Birkhoff Ergodic Theorem, for μ_1-almost every x, taking the limit as $n \to \infty$ in (5.80) gives,

$$\int_{\Lambda_g}\log|J^u g(x)|d\mu_1 \geq (h_{top}(\bar{f})-\epsilon)\mu_1(\Lambda_g\setminus B(0,r_1)) - \frac{\kappa - \log C_2}{Q}. \tag{5.81}$$

The construction of SRB measures presented in [3] as well as distortion estimates derived in Section (7.4) of [3] indicate, that there exists $L > 0$, not decreasing with r_1, such that for any $r > 0$, $\mu_1(B(0,r)) \leq Lr$. Consequently, as $r_1 \to 0$, the first term in (5.81) goes to $h_{top}(\bar{f}) - \epsilon > \frac{h_{top}(\bar{f})+h}{2} > h$. At the same time, as $r_1 \to 0$, then $Q \to \infty$, making the second term in (5.81) converge to zero. We conclude that if $r_1 > 0$ is small enough, then, $\int_{\Lambda_g}\log|J^u g(x)|d\mu_1 > h$. $\square$

By Theorem 4.2 and the above lemma, the map g satisfies the assertion of Statement 2 in Proposition 4.1 with some $t_0 < 0$. If in addition $|J^u f(x)|$ is a constant on Λ, using the arguments as in the proof of the second part of Lemma 5.13, one can show that $t_0 \to -\infty$ as $r_1 \to 0$.

To see that the map g satisfies the assertion of Statement 3 in Proposition 4.1 assume first that $gcd\{\tau_i \mid i \in \mathbb{N}\} = \kappa \neq 1$. Then the set

$$P \cup \bigcup_{m\geq 1}\ \bigcup_{\{i\ \mid\ \tau_i > m\kappa\}} g^{m\kappa}(P_i^s)$$

is invariant under g^κ. This contradicts the fact that g is topologically mixing on Λ. To show Condition (4.8) note that if $y \in W_x^s$, then by Remark 5.1 we have for all $j \geq 0$ that $d(f^j(x), f^j(y)) \leq C^2 d(x,y)$, where $C > 0$ comes from Proposition 3.2. On the other hand, if $F(y) \in W^u_{F(x)}$, then $d(F(x),F(y))$ $= d(f^{\tau(x)}(x), f^{\tau(y)}(y)) \geq C^2 d(f^j(x), f^j(y))$ for all $0 \leq j \leq \tau(x)$.

Applying Proposition 4.1 we obtain the existence of an equilibrium measure μ_t for $\varphi_{t,g}$, where $t \in (t_0, 1)$. In addition, μ_t is the unique equilibrium measure for $\varphi_{t,g}$ among all invariant measures giving positive weight to P. We now show that in fact μ_t is the unique equilibrium measure for $\varphi_{t,g}$ among all invariant measures. For this set $t \in (t_0, 1)$ and consider an equilibrium measure μ for $\varphi_{t,g}$. Assume first that $\mu(P') > 0$ for some $P' \in \mathcal{P}$ such that $P' \cap Z = \emptyset$ (see Condition (C4) for the definition of Z). Proposition 4.1 gives that there is only one such μ. It is obtained from a Gibbs measure for an induced map on P' (see [9]). In particular, it gives positive weight to every open subset of P'. Topological transitivity implies that $g^k(U) \subset P$ for some open set $U \subset P'$ and $k \in \mathbb{N}$. Then invariance of μ implies that $\mu(P) > 0$, so that $\mu = \mu_t$. Note that the only invariant measure which gives zero weight to the set $\Lambda_g \setminus \bigcup_{P' \in \mathcal{P}, P' \cap Z = \emptyset} P'$ is the Dirac measure at zero, which has zero pressure for all $t \in \mathbb{R}$. The claim follows by observing that $P(\varphi_{t,g}) > 0$ for all $t < 1$.

□

References

[1] L. Bareira, Y. Pesin, *Nonuniform hyperbolicity: dynamics of systems with nonzero Lyapunov exponents*, volume 115 of Encyclopedia of Mathematics and Its Applications, Cambridge University Press, New York, 2007

[2] R. Bowen, *Equilibrium states and the ergodic theory of Anosov diffeomorphisms*, revised ed., Lecture Notes in Mathematics, vol. 470, SpringerVerlag, Berlin, 2008, With a preface by David Ruelle, Edited by JeanRen'e Chazottes. MR 2423393

[3] V. Climenhaga, D. Dolgopyat, Y. Pesin, *Non-stationary non-uniform hyperbolicity*, Comm. Math. Phys., 346: 553. doi:10.1007/s00220-016-2710-z, 2016

[4] A. A. Gura, *Separating diffeomorphisms of the torus*, Math. Notes (18), 605-610, 1975

[5] B. Hasselblatt, *Ergodic Theory and Negative Curvature*, Springer Lecture Notes series, in preparation, 2017

[6] A. Katok, B. Hasselblatt, *Introduction to the modern theory of dynamical systems*, Volume 54 of Encyclopedia of Mathematics and its Applications, Cambridge University Press, 1995

[7] A. Katok, *Bernoulli diffeomorphisms on surfaces*, Ann. of Math. (2), 110(3):529-547, 1979

[8] I. Melbourne, D. Terhesiu, *Decay of correlations for non-uniformly expanding systems with general return times*, Ergodic Theory and Dynamical Systems, 34, pp 893-918 doi:10.1017/etds.2012.158, 2014

[9] Y. Pesin, S. Senti, K. Zhang, *Thermodynamics of towers of hyperbolic type*, Trans. Amer. Math. Soc., Soc. 368, 8519-8552, 2016

[10] Y. Pesin, S. Senti, K. Zhang, *Thermodyamics of the Katok map*, (preprint, arXiv:1603.08556), 2015

[11] L.-S. Young, *Statistical properties of dynamical systems with some hyperbolicity*, Ann. of Math. (2), 147(3):585-650, 1998

[12] L.-S. Young, *Recurrence times and rates of mixing*, Israel J. Math. 110, 153-188, 1999

Department of Mathematics, Pennsylvania State University, University Park, PA 16802, USA
E-mail address: `axz157@psu.edu`